Numerical and Symbolic Experimentis with the Harry Dym Equation

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In fond memory of Harry Dym (Jan. 26, 1938 - July 18, 2024)

Abstract: We use both symbolic and numeric computations to experiment with the Harry Dym equation, of central important in soliton theory.

Theorem: Let $a_0(t)$, $a_1(t)$, and $a_2(t)$ be *any* formal power series in t. Then the first 10 terms in the 'series expansion', in x, of the *general solution* $u(x,t)$, of the celebrated Harry Dym equation

$$u_t = u^3 u_{xxx}$$

in the ring of formal power series in (x,t) are

$$a_0(t) + a_1(t)x + a_2(t)x^2 + \frac{\frac{d}{dt}a_0(t)}{6a_0(t)^3}x^3 + \frac{\left(\frac{d}{dt}a_1(t)\right)a_0(t) - 3a_1(t)\left(\frac{d}{dt}a_0(t)\right)}{24a_0(t)^4}x^4$$

$$-\frac{3\left(\frac{d}{dt}a_1(t)\right)a_1(t)\,a_0(t) - 6a_1(t)^2\left(\frac{d}{dt}a_0(t)\right) - \left(\frac{d}{dt}a_2(t)\right)a_0(t)^2 + 3a_2(t)\,a_0(t)\left(\frac{d}{dt}a_0(t)\right)}{60a_0(t)^5}x^5$$

$$+\frac{x^6}{720a_0(t)^7}\cdot$$

$$(36\left(\frac{d}{dt}a_1(t)\right)a_1(t)^2\,a_0(t)^2 - 18\left(\frac{d}{dt}a_1(t)\right)a_0(t)^3\,a_2(t) - 60a_1(t)^3\,a_0(t)\left(\frac{d}{dt}a_0(t)\right) - 18a_1(t)\,a_0(t)^3\left(\frac{d}{dt}a_2(t)\right)$$

$$+72a_1(t)\,a_0(t)^2\,a_2(t)\left(\frac{d}{dt}a_0(t)\right) + \left(\frac{d^2}{dt^2}a_0(t)\right)a_0(t) - 6\left(\frac{d}{dt}a_0(t)\right)^2)$$

$$+\frac{x^7}{5040a_0(t)^8}\cdot$$

$$(1800\left(\frac{d}{dt}a_1(t)\right)a_1(t)^4\,a_0(t)^2 - 3600\left(\frac{d}{dt}a_1(t)\right)a_1(t)^2\,a_0(t)^3\,a_2(t)$$

$$+720\left(\frac{d}{dt}a_1(t)\right)a_0(t)^4\,a_2(t)^2 - 2520a_1(t)^5\,a_0(t)\left(\frac{d}{dt}a_0(t)\right) - 1200a_1(t)^3\,a_0(t)^3\left(\frac{d}{dt}a_2(t)\right)$$

$$+7200a_1(t)^3\,a_0(t)^2\,a_2(t)\left(\frac{d}{dt}a_0(t)\right)$$

$$+1440a_1(t)\,a_0(t)^4\left(\frac{d}{dt}a_2(t)\right)a_2(t) - 3600a_1(t)\,a_0(t)^3\,a_2(t)^2\left(\frac{d}{dt}a_0(t)\right) - 21a_1(t)\,a_0(t)^2\left(\frac{d^2}{dt^2}a_1(t)\right) + 2a_0(t)^3\left(\frac{d^2}{dt^2}\right.$$

$$+177a_1(t)^2\left(\frac{d^2}{dt^2}a_0(t)\right)a_0(t) - 1416a_1(t)^2\left(\frac{d}{dt}a_0(t)\right)^2 - 66\left(\frac{d^2}{dt^2}a_0(t)\right)a_0(t)^2\,a_2(t) - 78a_0(t)^2\left(\frac{d}{dt}a_2(t)\right)\left(\frac{d}{dt}a_0(\right.$$

$$+720\left(\frac{d}{dt}a_1(t)\right)a_0(t)^4\,a_2(t)^2-2520a_1(t)^5\,a_0(t)\left(\frac{d}{dt}a_0(t)\right)-1200a_1(t)^3\,a_0(t)^3\left(\frac{d}{dt}a_2(t)\right)+7200a_1(t)^3\,a_0(t)^2\,a_2(t$$

$$+1440a_1(t)\,a_0(t)^4\left(\frac{d}{dt}a_2(t)\right)a_2(t)-3600a_1(t)\,a_0(t)^3\,a_2(t)^2\left(\frac{d}{dt}a_0(t)\right)-21a_1(t)\,a_0(t)^2\left(\frac{d^2}{dt^2}a_1(t)\right)+2a_0(t)^3\left(\frac{d^2}{dt^2}\right.$$

$$+177a_1(t)^2\left(\frac{d^2}{dt^2}a_0(t)\right)a_0(t)-1416a_1(t)^2\left(\frac{d}{dt}a_0(t)\right)^2-66\left(\frac{d^2}{dt^2}a_0(t)\right)a_0(t)^2\,a_2(t)-78a_0(t)^2\left(\frac{d}{dt}a_2(t)\right)\left(\frac{d}{dt}a_0(\right.$$

References

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