The Number of Inversions and the Major Index of Permutations are Asymptotically Joint-Independently-Normal (First Edition)

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Abstract: We use *recurrences* (alias *difference equations*) to prove that the two most important *permutation statistics*, namely the *number of inversions* and the *major index*, are asymptotically joint-independently-normal. We even derive more-precise-than-needed asymptotic formulas for the (normalized) mixed moments.

Human Statistics

Human statistics are numerical attributes defined on *humans*, for example, *longevity*, *height*, *weight*, *IQ*, and it is well-known, at least empirically, that these are, each separately, *asymptotically normal*, which means that if you draw a histogram with the statistical data, it would look like a *bell-curve*. It is also true that they are usually joint-asymptotically-normal, but usually not *independently* so. But if you compute empirically the *correlation matrix*, you would get, asymptotically (i.e. for "large" populations) that they are close to being distributed according to a multivariate (generalized) *Gaussian* $exp(-Q(x_1, x_2, ...))$ with $Q(x_1, x_2, \dots)$ a certain *quadratic form* that can be deduced from the correlation matrix.

Permutation Statistics

Let our population be the set of *permutations* of $\{1, 2, \dots, n\}$. They too, can be assigned *numerical attributes*, and the great classical combinatorialist Dominique Foata (who got his 3rd-cycle doctorate in statistics!) coined the term *permutation statistics* for them.

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`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ChristianKrattenthalerInvMajReport.html`.
Accompanied by the Maple package `InvMaj` downloadable from
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/invmaj.html` .
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Exclusively published in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger
`http://www.math.rutgers.edu/~zeilberg/pj.html` . The second-named author is hereby offering \$1000 to the first person to point out a serious flaw in the argument, that would irreparably invalidate the proof.

The most important permutation statistic is the **number of inversions**, $inv(\pi)$, that counts the number of pairs $1 \leq i < j \leq n$ such that $\pi_i > \pi_j$ (and ranges from 0 to $n(n-1)/2$). For example, $inv(314625) = 5$, corresponding to the set of pairs $\{[1,2],[1,5],[3,5],[4,5],[4,6]\}$. It features in the definition of the determinant, and Netto proved that their *probability generating function* (the polynomial in q such that its coefficient of q^i is the probability that a uniformly-at-random n-permutation has i inversions) is given by

$$\frac{(1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\ldots+q^{n-1})}{n!} = \frac{\prod_{i=1}^{n}(1-q^i)}{n!(1-q)^n} \quad .$$

The *second most* important permutation statistic is the *major index*, $maj(\pi)$, that is the sum of the places i, where $\pi_i > \pi_{i+1}$. For example, $maj(314625) = 1+4 = 5$, because at $i=1$ and $i=4$ we have descents. Major Percy Alexander MacMahon [M] famously proved that the probability generating function for the major index is also given by that very same formula. In other words the permutation statistics *inv* and *maj* are *equidistributed*. Dominique Foata [Fo] gave a lovely seminal *bijective* proof that proved the stronger statement that *inv* and *maj* are equi-distributed also when restricted to permutations ending at a given integer.

William Feller ([Fe], 3rd ed., p.257) proved that the number of inversions (and hence also the major index) is *asymptotically normal* in the following sense. Feller easily computed the *expectation*,

$$E[inv] = m_n = n(n-1)/4 \quad ,$$

and the *variance*,

$$\sigma_n^2 = \frac{2n^3+3n^2-5n}{72} \quad .$$

If we denote by $\mathcal{Z}_n$ the *centralized* and *normalized* random variable

$$\mathcal{Z}_n = \frac{inv - m_n}{\sigma_n} \quad ,$$

then $\mathcal{Z}_n \rightarrow \mathcal{N}$, as $n \rightarrow \infty$, in distribution, where $\mathcal{N}$ is the Gaussian distribution whose *probability density function* is $e^{-x^2/2}/\sqrt{2\pi}$.

A *computer-generated* proof, that gives much more detail about the rate of convergence to $\mathcal{N}$, can be obtained using Zeilberger's Maple package
`http://www.math.rutgers.edu/~zeilberg/tokhniot/AsymptoticMoments` that accompanies the article [Z].

So both *inv* and *maj* are *individually* asymptotically normal, but what about their *interaction*, in other words, what can you say about the limit of

$$\frac{1}{da\,db} Pr(\sigma_n a \leq inv(\pi) - m_n \leq \sigma_n(a+da) \quad AND \quad \sigma_n b \leq maj(\pi) - m_n \leq \sigma_n(b+db) \quad) \quad ,$$

as $n \rightarrow \infty$ and $da, db \rightarrow 0$?

In this article, we prove that this limit exists and equals $(2\pi)^{-1}e^{-a^2/2-b^2/2}$. In other words, *inv* and *maj* are asymptotically *joint-independently-normal.*

A Brief History

It all started when the great Swedish probabilist Svante Janson (member of the Swedish Academy of Science, that awards the Nobel prizes) asked Donald Knuth (one of the greatest computer scientists of all time, winner of the Turing and Kyoto prizes, among many other honors) about the *asymptotic covariance* of *inv* and *maj*. Neither of these luminaries knew the answer, so Don Knuth asked one of us (DZ). DZ didn't know the answer either, so he asked his beloved servant, Shalosh B. Ekhad, who *immediately* ([E]) produced, not just the asymptotics, but the *exact answer*! It turned out to be $n(n-1)/8$. In particular, the *correlation*, $Cov(inv, maj)/\sigma_n^2 = \frac{\frac{n(n-1)}{8}}{\frac{2n^3+3n^2-5n}{72}} = \frac{9}{2n} + O(1/n^2)$ tends to zero as n goes to infinity. It followed that in the long-run, *inv* and *maj* are practically *uncorrelated.*

But there are lots of pairs of random variables that are uncorrelated yet not independent. A convenient way to prove that $X_n := (inv - m_n)/\sigma_n$ and $Y_n := (maj - m_n)/\sigma_n$ are asymptotically independent (we already know that they are both normal) is to use the method of moments, and to prove that the *mixed moments*

$$M_{r,s}(n) := E[X_n^r Y_n^s] \quad ,$$

tend to the mixed moments of $\mathcal{N} \times \mathcal{N}$, as $n \to \infty$. In other words, for $r, s \geq 1$:

$$\lim_{n\to\infty} M_{2r,2s}(n) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!} \quad , \tag{EE}$$

$$\lim_{n\to\infty} M_{2r-1,2s}(n) = 0 \quad , \tag{OE}$$

$$\lim_{n\to\infty} M_{2r,2s-1}(n) = 0 \quad , \tag{EO}$$

$$\lim_{n\to\infty} M_{2r-1,2s-1}(n) = 0 \quad . \tag{OO}$$

Ekhad's brilliant approach to the Janson-Knuth question merely settled the case $r = 1, s = 1$ of (OO). Of course, because of symmetry (OE) and (EO) are trivially true (before taking the limits!, i.e. $M_{2r,2s-1}(n) \equiv 0$ and $M_{2r-1,2s}(n) \equiv 0$).

One natural approach would be to extend Ekhad's brilliant answer for $M_{1,1}(n)$ to the general case, and try to derive closed-form expressions for $M_{r,s}(n)$ for larger r and s. Since Ekhad's proof [E] is so brief, we can cite it here in full.

"Svante Janson asked Don Knuth, who asked me, about the covariance of *inv* and *maj*. The answer is $\binom{n}{2}/4$. To prove it, I asked Shalosh to compute the average of the quantity $(inv(\pi) - E(inv))(maj(\pi) - E(maj))$ over all permutations of a given length n, and it gave me, for $n =$

$1, 2, 3, 4, 5$, the values $0, 1/4, 3/4, 3/2, 5/2$, respectively. Since we know a priori[2] that this is a polynomial of degree ≤ 4, this must be it! $\square$".

Obviously this brute-brute-force approach would be hopeless for deriving polynomial expressions for the moments $M_{r,s}(n)$ for larger r and s. As we will soon see, the degree of the polynomial $M_{2r,2s}(n)$ is $3(r+s)$, so for example, in order to (rigorously) guess $M_{10,10}(n)$, we would need 31 data points, and we would have to ask our computers to examine more than $31! > 0.822 \cdot 10^{34}$ permutations.

However, an *inspired*, still "empirical" (yet **fully rigorous**) "brute-force" approach *does* work. The first step would be to have a more efficient way to compute the moments $M_{r,s}(n)$, for *specific* n and *specific* r and s. We will do it by first designing an efficient way to generate the *probability generating function*, let's call it $G(n)(p, q)$, for the pair of statistics (inv, maj). There are beautiful closed-form expressions for $G(n)(1, q)$ and $G(n)(p, 1)$ (given above), but no such closed-form expression seems to exist for the bi-variate generating function, so the best that we can hope for is to find a *recurrence scheme*.

A Combinatorial Interlude

Let us forget about probability for a few moments, and focus on a fast algorithm for computing

$$H(n)(p, q) := \sum_{\pi \in S_n} p^{inv(\pi)} q^{maj(\pi)} \quad ,$$

for n up to, say, $n = 50$.

Christian Krattenthaler, in his referee report mentioned in footnote 1, believes that we should mention, at this point, the seminal work of Adriano Garsia and Ira Gessel [GG] on permutation statistics, that may perhaps offer an alternative approach for proving the main result of the present article, but this remains to be seen.

Define the *weight* of a permutation π to be $p^{inv(\pi)} q^{maj(\pi)}$. Suppose that $\pi \in S_n$ ends with i, so we can write $\pi = \pi' i$, where π' is a permutation of $\{1, \dots, i-1, i+1, \dots n\}$.

When you chop off i from π to form π' you always lose $n - i$ inversions (that is, π' has $n - i$ fewer inversions than π). The major index, however, decreases by $n - 1$ if the last letter of π', let's call it j, is larger than i. If $j < i$ the major index does not change at all. So writing $\pi = \pi'' j i$, we have

$$inv(\pi'' j i) = inv(\pi'' j) + n - i \quad ,$$

$$maj(\pi'' j i) = \begin{cases} maj(\pi'' j), & \text{if } j < i \ ; \\ maj(\pi'' j) + n - 1, & \text{if } j > i. \end{cases}$$

[2] This is the old trick to compute moments of combinatorial 'statistics', described nicely [GKP], section 8.2, by changing the order of summation. It applies equally well to covariance. Rather than actually carrying out the gory details, we observe that this is always a polynomial whose degree is trivial to bound.

Combining, we have

$$weight(\pi''ji) = \begin{cases} p^{n-i} weight(\pi''j), & \text{if } j < i \text{ ;} \\ p^{n-i}q^{n-1} weight(\pi''j), & \text{if } j > i. \end{cases}$$

So in order to compute $H(n)(p,q)$, we need to introduce the more general weight-enumerators of those permutations in S_n that end with an i. Let's call these $F(n,i)(p,q)$. In symbols:

$$F(n,i)(p,q) := \sum_{\substack{\pi \in S_n \\ \pi_n = i}} p^{inv(\pi)} q^{maj(\pi)} \quad .$$

It follows that (let's omit the arguments (p,q) from now on):

$$F(n,i) = p^{n-i} \sum_{j=1}^{i-1} F(n-1,j) + p^{n-i}q^{n-1} \sum_{j=i}^{n-1} F(n-1,j) \quad . \qquad (Fni)$$

Note that, when we chop off the last entry, i, from $\pi = \pi''ji$, $\pi''j$ is a permutation of $\{1, \ldots, i-1, i+1, \ldots, n\}$. We then "reduce" $\pi''j$ to a permutation of $\{1, \ldots, n-1\}$ by diminishing all entries larger than i by 1. Hence the summation ranges from $j = i$ to $j = n-1$ rather than from $j = i+1$ to $j = n$.

Replacing i by $i+1$ in the above equation, we have:

$$F(n,i+1) = p^{n-i-1} \sum_{j=1}^{i} F(n-1,j) + p^{n-i-1}q^{n-1} \sum_{j=i+1}^{n-1} F(n-1,j) \quad .$$

Subtracting the former equation from p times the latter we get

$$F(n,i) - pF(n,i+1) =$$

$$p^{n-i} \sum_{j=1}^{i-1} F(n-1,j) + p^{n-i}q^{n-1} \sum_{j=i}^{n-1} F(n-1,j)$$

$$-p^{n-i} \sum_{j=1}^{i} F(n-1,j) - p^{n-i}q^{n-1} \sum_{j=i+1}^{n-1} F(n-1,j)$$

$$= p^{n-i} \left(\sum_{j=1}^{i-1} F(n-1,j) - \sum_{j=1}^{i} F(n-1,j) \right) + p^{n-i}q^{n-1} \left(\sum_{j=i}^{n-1} F(n-1,j) - \sum_{j=i+1}^{n-1} F(n-1,j) \right)$$

$$= -p^{n-i}F(n-1,i) + p^{n-i}q^{n-1}F(n-1,i) = p^{n-i}(q^{n-1}-1)F(n-1,i) \quad .$$

Rearranging, we get:

$$F(n,i) = pF(n,i+1) + p^{n-i}(q^{n-1}-1)F(n-1,i) \quad . \qquad (RecF)$$

We still need to specify $F(n,n)$, and for this we do need the $\sum$ symbol, namely we use Eq. (Fni) with $i = n$:

$$F(n,n) = \sum_{j=1}^{n-1} F(n-1,j) \quad . \qquad (Fnn)$$

The recurrence $(RecF)$ together with the final condition (Fnn), and the trivial *initial condition* $F(1,i) = \delta_{i,1}$, enables us to efficiently compute $F(n,i)$ for numeric (n,i), for $\{(n,i) | 1 \le i \le n \le N\}$ for any finite N (not too large, but not too small either: e.g., $N = 100$ is still feasible). In particular, we can compile a table of $H(n)(p,q) = F(n+1,n+1)(p,q)$, for $n \le N-1$.

A crash course in multivariable enumerative probability

Suppose that you have a finite set of objects S and several *statistics* $f_1(s), \dots, f_r(s)$. The *multivariable generating function* (weight-enumerator under the weight $x_1^{f_1(s)} \cdots x_r^{f_r(s)}$) is defined to be:

$$\sum_{s \in S} x_1^{f_1(s)} \cdots x_r^{f_r(s)} \quad .$$

Suppose that you pick an element $s \in S$ *uniformly at random* and you want the multivariable generating function such that the coefficient of $x_1^{a_1} \cdots x_r^{a_r}$ would give you the probability that $f_1(s) = a_1, \dots, f_r(s) = a_r$. It is given by:

$$P(x_1, \dots, x_r) = \frac{1}{|S|} \sum_{s \in S} x_1^{f_1(s)} \cdots x_r^{f_r(s)} \quad .$$

The *expectations*, $\bar{f}_1, \dots, \bar{f}_r$ are simply

$$\bar{f}_i = \left(\frac{\partial}{\partial x_i} P \right)(1, \dots, 1) \quad .$$

The *centralized* probability generating function is

$$\tilde{P}(x_1, \dots, x_r) = \frac{P(x_1, \dots, x_r)}{x_1^{\bar{f}_1} \cdots x_r^{\bar{f}_r}} \quad .$$

The mixed moments (about the mean) of the statistics $f_1(s), \dots, f_r(s)$ are defined by

$$Mom[a_1, \dots, a_r] = \frac{1}{|S|} \sum_{s \in S} (f_1(s) - \bar{f}_1)^{a_1} \cdots (f_r(s) - \bar{f}_r)^{a_r} \quad .$$

Often it is more convenient to consider the mixed *factorial moments*, using the combinatorial "powers" $z^{(r)} := z(z-1)\cdots(z-r+1)$, better known as the *falling-factorials*. The mixed factorial moments are defined analogously by

$$FM[a_1, \dots, a_r] = \frac{1}{|S|} \sum_{s \in S} (f_1(s) - \bar{f}_1)^{(a_1)} \cdots (f_r(s) - \bar{f}_r)^{(a_r)} \quad .$$

Once you know the FM's for all $a_1, \dots, a_r \leq M$, you can easily figure out the Mom's (for $a_1, \dots, a_r \leq M$), using Stirling numbers of the second kind (see [Z]). In particular, for the present purposes, where our finite sets are the symmetric groups S_n, the leading terms of the FM's and the Mom's are the same, so for the leading asymptotics, it suffices to consider the easier FM's.

The best way to compute $FM[a_1, \dots, a_r]$ for all $0 \leq a_1, \dots, a_r \leq M$, for some fixed positive integer M, is via the *Taylor expansion* of $\tilde{P}(x_1, \dots, x_r)$ around $(x_1, \dots, x_r) = (1, \dots, 1)$, or equivalently, the *Maclaurin expansion* of $\tilde{P}(1 + x_1, \dots, 1 + x_r)$

$$\tilde{P}(1 + x_1, \dots, 1 + x_r) = \sum_{\alpha_1, \dots, \alpha_r \geq 0} \frac{FM(\alpha_1, \dots, \alpha_r)}{\alpha_1! \cdots \alpha_r!} x_1^{\alpha_1} \cdots x_r^{\alpha_r} \quad .$$

Back to inv-maj

In the present approach, we need to put up with the more general discrete function $F(n,i)(p,q)$ even though ultimately we are only interested in $H(n)(p,q) = F(n+1,n+1)(p,q)$. We will prove the stronger statement that even if you restrict attention to those $(n-1)!$ permutations that end with a specific i, it is still asymptotically-joint-independently normal.

Since the averages of both *inv* and *maj* over the permutations that end in i is $n-i+(n-1)(n-2)/4$, the *centralized probability generating function* corresponding to $F(n,i)(p,q)$ is:

$$G(n,i)(p,q) := \frac{F(n,i)(p,q)}{(n-1)!(pq)^{n-i+(n-1)(n-2)/4}} \quad .$$

The recurrence ($RecF$) becomes

$$G(n,i) = \frac{1}{q}G(n,i+1) + \frac{p^{n-i}(q^{n-1}-1)}{(pq)^{n/2}(n-1)}G(n-1,i) \quad , \qquad (RecG)$$

and the *final condition* becomes

$$G(n,n) = \frac{1}{n-1}\sum_{j=1}^{n-1}(pq)^{n/2-j}G(n-1,j) \quad . \qquad (Gnn)$$

We also need the obvious *initial condition* $G(1,i) = \delta_{i,1}$.

Guessing Polynomial Expressions for the Factorial Moments

Equipped with these efficient recurrences, our computer computes $G(n,i)(p,q)$ for many values of n and i. Then for each of these it computes the factorial moments $FM(r,s)(n,i)$, for many small numeric (r,s) by computing the (initial terms of the) Maclaurin series for $G(n,i)(1+p,1+q)$. We then fix numeric values of (r,s) and use *polynomial interpolation* to *guess* explicit polynomial expressions for $FM(r,s)(n,i)$ as polynomials in (n,i) for that pair (r,s). The process is repeated for all pairs (r,s) for which $0 \le r,s \le M$, for some pre-determined specific positive integer M. We note that it is obvious, both from the combinatorics and from the recurrences, that the $FM(r,s)(n,i)$ are always polynomials in (n,i), for any fixed numeric r and s.

It would have been nice if we could guess *closed-form* expressions for $FM(r,s)(n,i)$ for *symbolic* (r,s), but no such closed-form exists as far as we know, and besides it is too much to ask for and more than we need. To prove *asymptotic normality* we only need the *leading terms*. Viewing the leading terms, our beloved computer easily conjectures the following expressions, For integers $r \ge 0, s \ge 0$ we have:

$$FM(2r,2s)(n,i) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(\frac{1}{36}\right)^{r+s} n^{3r+3s} + (lower-degree-terms-in-(n,i)) \quad .$$

For integers $r \ge 0, s \ge 1$ we have:

$$FM(2r,2s-1)(n,i) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(\frac{1}{36}\right)^{r+s-1} n^{3r+3s-6}[-(s-1)n^3 - 6rn^2i + 18rni^2 - 12ri^3]$$

$$+(lower-degree-terms-in-(n,i)) \quad .$$

For integers $r \geq 1, s \geq 0$ we have:

$$FM(2r-1,2s)(n,i) = -\frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(\frac{1}{36}\right)^{r+s-1}(r-1)n^{3r+3s-3}+(lower-degree-terms-in-(n,i)) \quad .$$

For integers $r \geq 1, s \geq 1$ we have:

$$FM(2r-1,2s-1)(n,i) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(\frac{1}{36}\right)^{r+s-1}\frac{9}{2}n^{3r+3s-6}(n-2i)^2+(lower-degree-terms-in-(n,i)) \quad .$$

Nice conjectures but what about proofs?

While we prefer the empirical approach of guessing, an alternative approach to finding many $FM(r,s)(n,i)$'s is to first use (*RecG*) and (*Gnn*). Write $G(n,i)(1+p,1+q)$ as an *infinite* **generic** Taylor series around $(0,0)$, and write-down the implied *infinite-order* recurrences expressing $FM(r,s)$ in terms of $FM(r',s')$ with $r'+s' < r+s$. Note that in order to compute $FM(r,s)$ we only need finitely many terms of the infinite-order recurrence, since eventually all the contributions will be zero. Of course, as we have already commented, there is no hope for finding a general expression for $FM(2r,2s)(n,i)$, $FM(2r,2s-1)(n,i)$, $FM(2r-1,2s)(n,i)$ and $FM(2r-1,2s-1)(n,i)$, depending explicitly on r and s as well as on n and i, but to prove, by induction on r,s, that the above *leading terms* are valid, all we need to do is to verify that the leading terms of the implied recurrences for the $FM(r,s)$'s (easily derivable by hand, although we used the computer) are consistent with the above explicit expressions.

The implication of (*RecG*) is

$$FM(r,s)(n,i)-FM(r,s)(n,i+1) = -sFM(r,s-1)(n,i+1)+sFM(r,s-1)(n-1,i)+(lower-order-terms) \quad , \tag{RecG'}$$

while the implication of *Gnn* is:

$$FM(r,s)(n,n) = \frac{1}{n-1}\sum_{i=1}^{n-1} FM(r,s)(n-1,i) - \frac{s}{2(n-1)}\sum_{i=1}^{n-1}(2i-n)FM(r,s-1)(n-1,i) -$$

$$\frac{r}{2(n-1)}\sum_{i=1}^{n-1}(2i-n)FM(r-1,s)(n-1,i) + \frac{rs}{4(n-1)}\sum_{i=1}^{n-1}(2i-n)^2 FM(r-1,s-1)(n-1,i)$$

$$+(lower-order-terms) \quad . \tag{Gnn'}$$

The next step is to spell-out these two recurrences each into the four cases according to whether (r,s) is $(even, even)$, $(even, odd)$, $(odd, even)$, and (odd, odd).

Once you have these eight recurrences, for each and every one of them, you plug-in the above conjectured expressions for the leading terms, and verify that up to the leading terms, things agree. At the end of the day, after dividing by

$$\frac{(2r-2)!}{(r-1)!2^{r-1}}\frac{(2s-2)!}{(s-1)!2^{s-1}} \quad ,$$

this boils down to proving equalities among certain low-degree polynomials in (r,s) (namely the leading coefficients, in (n,i)), that in turn, reduces (since $A=B$ iff $A-B=0$) to proving that certain low-degree polynomials in (r,s) are identically zero.

So in order to check that these low-degree polynomials in (r,s) are all identically zero, it is enough to check each of them for finitely (and not-too-many) *numeric* r,s. Typing `Check1(FM8m(n,i),n,i);` and `Check2(FM8m(n,i),n,i);` in the Maple package `InvMaj` does exactly that, by checking that if you plug-in the conjectured leading terms of $FM(2r,2s)(n,i)$, $FM(2r-1,2s)(n,i)$, $FM(2r,2s-1)(n,i)$ and $FM(2r-1,2s-1)(n,i)$ and subtract the right sides from the left sides (for each of the eight cases) you get lower-order polynomials, in (n,i), for all $1 \leq r,s \leq 8$. This proves all these claims (rigorously), with a vengeance! The $(8/2)^2=16$ special cases are much more than is needed, since the relevant polynomials in (r,s) are easily seen to have degree ≤ 2 so $(2+1)^2=9$ agreements would have sufficed.

The Maple package InvMaj

All the nitty-gritty calculations described above, that constitute **a fully rigorous** proof, may be found in the Maple package `InvMaj` accompanying this article. This package is available from the webpage of the present article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/invmaj.html` ,

where the reader can also find some sample input and output. The direct url of the package is:
`http://www.math.rutgers.edu/~zeilberg/tokhniot/InvMaj` .

La Grande Finale

The special case $r=1, s=0$ and $r=0, s=1$ give

$$FM(2,0)(n,i) = \frac{1}{36}n^3 + O(n^2) \quad ,$$

$$FM(0,2)(n,i) = \frac{1}{36}n^3 + O(n^2) \quad .$$

So

$$\frac{FM(2r,2s)}{FM(2,0)^r FM(0,2)^s} = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!} + O(1/n) \quad ,$$

$$\frac{FM(2r,2s-1)}{FM(2,0)^r FM(0,2)^{s-1/2}} = o(1/n) \quad ,$$

$$\frac{FM(2r-1,2s)}{FM(2,0)^{r-1/2} FM(0,2)^{s}} = o(1/n) \quad ,$$

$$\frac{FM(2r-1,2s-1)}{FM(2,0)^{r-1/2} FM(0,2)^{s-1/2}} = O(1/n) \quad .$$

And we see that as $n \to \infty$ these indeed converge to the mixed moments of the famous mixed moments of the bivariate independent normal distribution $e^{-a^2/2-b^2/2}/(2\pi)$ $\square$.

Encore: A more refined asymptotics for the (Normalized) Mixed Moments

With more effort, we (or rather, our computer) can guess-and-prove the following asymptotics for the case of interest $(n+1, n+1)$, i.e. the asymptotic expressions for the centralized-and-normalized (genuine, not factorial) mixed-moments, let's call them $\alpha(r,s)(n)$, for the pair of random variables (inv, maj) acting on the set of permutations of length n. Indeed, according to S. B. Ekhad, we have:

$$\alpha(2r,2s)(n) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(1 - \frac{9(r^2+s^2-r-s)}{25}\cdot\frac{1}{n} + O(\frac{1}{n^2})\right) \quad ,$$

$$\alpha(2r-1,2s-1)(n) = \frac{(2r)!}{2^r r!}\frac{(2s)!}{2^s s!}\left(\frac{9}{2n} + (-\frac{81}{50}(r^2+s^2) + \frac{243}{50}(r+s) - \frac{1773}{100})\frac{1}{n^2} + O(\frac{1}{n^3})\right) \quad .$$

Of course, by *symmetry* $\alpha(2r, 2s-1)(n)$ and $\alpha(2r-1, 2s)(n)$ are *identically* (not just asymptotically!) zero.

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