On Rademacher's Infinite Partial Fraction Conjecture

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"It is a capital mistake to theorise before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."—Sherlock Holmes to Dr. Watson in *A Scandal in Bohemia* by A. Conan Doyle, 1891.

Important Note: This article is accompanied by the Maple package `HANS`, downloadable from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/HANS`.

The "front" of this article,

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hans.html`
contains lots of supporting input and output files.

1 Introduction

Let $p_N(n)$ denote the number of partitions of the integer n into at most N parts. The generating function of $p_N(n)$,

$$F_N(x) := \sum_{n\geq 0} p_N(n)x^n = \prod_{j=1}^{N} \frac{1}{1-x^j},$$

may be decomposed into partial fractions:

$$\prod_{j=1}^{N} \frac{1}{1-x^j} = \sum_{k=1}^{N} \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \sum_{l=1}^{\lfloor N/k \rfloor} \frac{C_{h,k,l}(N)}{(x-e^{2\pi i h/k})^l}. \tag{1.1}$$

We shall refer to the $C_{h,k,l}(N)$ defined by (1.1) as the *Rademacher coefficients*.

Near the end of his posthumously published masterpiece *Topics in Analytic Number Theory* [4, p. 302], Hans Rademacher made the following conjecture:

Rademacher's Conjecture. *For all integers h, k, l such that $0 \leq h < k$, $gcd(h,k) = 1$ and $l \geq 1$, $\lim_{N\to\infty} C_{h,k,l}(N)$ exists and equals*

$$R_{h,k,l} := -2\pi \left(\frac{\pi}{12}\right)^{3/2} \frac{e^{\pi i (s(h,k)+2hl/k)}}{k^{5/2}} \Delta_\alpha^{l-1} L_{3/2}\left(-\frac{\pi^2}{6k^2}(\alpha+1)\right), \tag{1.2}$$

evaluated at $\alpha = \frac{1}{24}$, where $s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \lfloor\frac{\mu}{k}\rfloor - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \lfloor\frac{h\mu}{k}\rfloor - \frac{1}{2}\right)$ is the Dedekind sum, Δ_α is the forward difference operator, so that

$$\Delta_\alpha^j f(\alpha) = \sum_{h=0}^{j} (-1)^h \binom{j}{h} f(\alpha+j-h),$$

and

$$L_{3/2}(-y^2) = -\frac{1}{2\sqrt{\pi}y^2}\left(2\cos(2y) - \frac{\sin(2y)}{y}\right).$$

In particular, if the Rademacher conjecture would have been true then it would have followed that

$$\lim_{N\to\infty} C_{0,1,1}(N) = R_{0,1,1}(= -\frac{6}{25}\left(1 + \frac{2\sqrt{3}}{5\pi}\right) = -0.292927573960\ldots),$$

$$\lim_{N\to\infty} C_{0,1,2}(N) = R_{0,1,2}(= \frac{144}{1225} + \frac{5616}{42875\pi} = 0.1897670688440\ldots),$$

$$\lim_{N\to\infty} C_{1,2,1}(N) = R_{1,2,1}(= -\frac{2\sqrt{6}}{25}\left(\cos\frac{5\pi}{12} - \frac{12}{5\pi}\sin\frac{5\pi}{12}\right) = 0.093882853484\ldots).$$

(Let us make the minor remark, for the sake of mathematical accuracy, that the floating-point approximation for the value of $R_{0,1,1}$ stated by Rademacher [4, p. 302] was erroneous, as were the exact values of $R_{0,1,2}$ and $R_{1,2,1}$ [for the latter he gave exactly one half of the correct value]. These erroneous values were quoted, without correction, by Andrews [1, p. 388].)

Rademacher supplied (with one error) a table of values for $C_{0,1,1}(N)$, $C_{0,1,2}(N)$, and $C_{1,2,1}(N)$ for $N = 1, 2, 3, 4, 5$, and in fact these values are not too far off from conjectured "$N = \infty$" case.

Rademacher began work on the book in which this conjecture appeared [4] no later than 1944 and was still working on it at the inception of his final illness. Thus as the final version was edited and published by Rademacher's students Emil Grosswald, Joseph Lehner, and Morris Newman, after Rademacher's death in 1969, we will never know whether Rademacher came to doubt the truth of the conjecture after he had written it down. However, George Andrews reports that Rademacher discussed the conjecture in a course he taught at the University of Pennsylvania in the 1961–1962 academic year.

In [3], Augustine Munagi considered an different type of partial fraction decomposition called q-partial fractions, and proved a special case of the analog of the Rademacher conjecture, relative to the q-partial fraction decomposition.

We should also note that the first to cast doubts on the Rademacher conjecture were Jane Friedman and Leon Ehrenpreis. Ehrenpreis [2, p. 317] stated, "If one attempts to carry out the usual type of partial fraction decomposition of the partition function term-by-term, it is difficult to compute the coefficients. My student, Jane Friedman, spent a great deal of time trying to apply computer algorithm methods to compare the coefficients with those of Rademacher... Unfortunately, the computer study proved inconclusive."

In this article, we present overwhelming evidence against this conjecture, taken literally, but we will present ample evidence for a modified conjecture. Additionally, we will present a fast algorithm for generating the Rademacher coefficients, and formulas for a selection of particular Rademacher coefficients.

2 Empirical evidence against the Rademacher conjecture

2.1 The *actual* behavior of the sequence $C_{0,1,1}(N)$

The values of $C_{0,1,1}(N)$, for $1 \le N \le 400$ are provided, in both exact rational and floating point approximation form, at
`http://www.math.rutgers.edu/ zeilberg/tokhniot/oHANS1`

Figures 1 and 2 are graphical summaries of $C_{0,1,1}(N)$.

Figure 1: Graph of $C_{0,1,1}(N)$ for N from 1 to 100, together with the line $y = -0.2929276$

Figure 2: Graph of $C_{0,1,1}(N)$ for N from 1 to 200.

Notice that $C_{0,1,1}(25)$ differs from Rademacher's conjectured value of $C_{0,1,1}(\infty)$ by less than 0.000032, but then things go down hill (for the conjecture) from there, and get particularly bad after about $n = 150$. Numerical evidence points to the sequence $C_{0,1,1}(N)$ oscillating and attaining arbitrarily large positive and negative values. The same thing is true for other sequences $C_{h,k,l}(N)$; see the output at the webpage of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/hans.html`

By examining the graphs in Figures 1 and 2 and the associated numerical data, it seems reasonable to state the following alternative conjecture:

Conjecture 1. *$C_{0,1,1}(N)$ is an oscillating function of N of "period" about 32, whose absolute value increases without bound as N increases.*

By "period" 32 we mean that the peaks and valleys, eventually, recur at a period of 32. We also noticed, numerically, that the elevations and depths of successive peaks and valleys roughly grows exponentially with a factor around 8.

2.2 How to compute the sequence $C_{0,1,1}(N)$ fast

If you use the **definition** of $C_{0,1,1}(N)$, or Andrews's formula [1, p. 388, Theorem 1], you can't go very far, even with Maple. Rademacher calculated $C_{0,1,1}(N)$ for $N = 1, 2, 3, 4, 5$, presumably by hand, and made an error in the $N = 5$ case. Andrews [1, p. 388], who had access to a computer algebra system in 2003, corrected Rademacher's error at $N = 5$ and extended the list to $N = 6, 7, 8$.

We need to be more clever. A fast recurrence can be derived as follows. Since

$$\prod_{j=1}^{N} \frac{1}{1-x^j} = \sum_{l=1}^{N} \frac{C_{0,1,l}(N)}{(x-1)^l} + \dots \quad ,$$

we can multiply both sides by $(x-1)^N$ and get

$$(x-1)^N \prod_{j=1}^{N} \frac{1}{1-x^j} = \sum_{r=0}^{N-1} D_r(N)(x-1)^r + \dots \quad .$$

Once we know $D_r(N)$, we can find $C_{0,1,l}(N)$, since they equal $D_{N-l}(N)$. It remains to find a fast recurrence for $D_r(N)$.

By definition, we have:

$$\frac{1-x^N}{x-1}\left(\sum_{r=0}^{\infty} D_r(N)(x-1)^r\right) = \sum_{r=0}^{\infty} D_r(N-1)(x-1)^r \quad .$$

Letting $z = x - 1$, this is:

$$\frac{1-(z+1)^N}{z}\left(\sum_{r=0}^{\infty} D_r(N)z^r\right) = \sum_{r=0}^{\infty} D_r(N-1)z^r \quad .$$

By the binomial theorem,

$$-\left(\sum_{a=0}^{N-1}\binom{N}{a+1}z^a\right)\left(\sum_{r=0}^{\infty} D_r(N)z^r\right) = \sum_{r=0}^{\infty} D_r(N-1)z^r \quad .$$

Equating coefficients of z^r we get:

$$N D_r(N) + \sum_{a=1}^{r} \binom{N}{a+1} D_{r-a}(N) = -D_r(N-1) \quad .$$

And finally:

$$D_r(N) = -\frac{D_r(N-1)}{N} - \sum_{a=1}^{r} \frac{1}{N} \binom{N}{a+1} D_{r-a}(N) \quad .$$

This is implemented in procedure `C01` of `HANS`.

The same argument leads to efficient recurrences for $C_{h,k,l}(N)$, except that now we have to distingish between the case when N is divisible by k and when it is not, yielding two different recurences. This is implemented in procedure `ChklN(h,k,l,N)` of `HANS`.

2.3 $C_{1,2,1}(N)$

The values of $C_{1,2,1}(N)$ for $1 \leq N \leq 700$, in both exact rational form and approximate floating point form are provided at

`http://www.math.rutgers.edu/ zeilberg/tokhniot/oHANS3`

Graphical summaries are provided in Figures 3 and 4.

Figure 3: Graph of $C_{1,2,1}(N)$ for N from 1 to 150, together with the line $y = 0.09388285$

Figure 4: Graph of $C_{1,2,1}(N)$ for N from 1 to 300, together with the line $y = 0.09388285$

3 "Top down" formulas for the Rademacher Coefficients

3.1 $C_{0,1,l}(N)$

As Rademacher already pointed out, it seems hopeless to get a closed-form formula for $C_{h,k,l}(N)$ for $l = 1, 2, \ldots,$ *but* if you work your way up from the "top" one can conjecutre, and then *rigorously* prove explicit formuals for $C_{h,k,N-r}(N)$, that alas, get increasingly more complicated as r gets larger.

Conjecture 2.

$$C_{0,1,N-r}(N) = \frac{(-1)^{N+r}}{4^r N! r!} P_{0,1,N-r}(N),$$

where $P_{0,1,N-r}(N)$ is a convex, alternating, monic polynomial of degree $2r$ whose only real roots are 0 *and* 1.

Theorem 3. *Explicit formulas for the $P_{0,1,N-r}(N)$ of Conjecture 2 may be given for any specific r, in particular, we have*

$$P_{0,1,N}(N) = 1 \tag{3.1}$$

$$P_{0,1,N-1}(N) = N^2 - N \tag{3.2}$$

$$P_{0,1,N-2}(N) = N^4 - \frac{22N^3}{9} + \frac{13N^2}{3} - \frac{26N}{9} \tag{3.3}$$

$$P_{0,1,N-3}(N) = N^6 - \frac{13N^5}{3} + \frac{43N^4}{3} - 25N^3 + \frac{98N^2}{3} - \frac{56N}{3} \tag{3.4}$$

$$P_{0,1,N-4}(N) = N^8 - \frac{20N^7}{3} + \frac{862N^6}{27} - \frac{21104N^5}{225} + \frac{29039N^4}{135} - \frac{14548N^3}{45} + \frac{9892N^2}{27} - \frac{42896N}{225}$$

$$\begin{aligned}P_{0,1,N-5}(N) = N^{10} - \frac{85N^9}{9} + \frac{1610N^8}{27} - \frac{33862N^7}{135} + \frac{36869N^6}{45} - \frac{269371N^5}{135} + \frac{34712N^4}{9} \\ - \frac{716548N^3}{135} + \frac{742928N^2}{135} - \frac{120448N}{45}\end{aligned}$$

$$\begin{aligned}P_{0,1,N-6}(N) = N^{12} - \frac{38N^{11}}{3} + \frac{905N^{10}}{9} - \frac{675574N^9}{1215} + \frac{321073N^8}{135} - \frac{158916134N^7}{19845} \\ + \frac{4121695N^6}{189} - \frac{134142298N^5}{2835} + \frac{11129492N^4}{135} - \frac{908622824N^3}{8505} \\ + \frac{2805920N^2}{27} - \frac{317019904N}{6615}\end{aligned}$$

Remark 4. Readers desiring formulas for $C_{0,1,N-r}(N)$ for $r > 6$ are directed to the `ChkFormula` procedure in the `HANS` Maple package.

Remark 5. From the above (and additional data not reproduced here but available at the website), we may deduce that

$$P_{0,1,N-r}(N) \quad = \quad N^{2r} \quad - \quad \frac{2r^2+7r}{9}N^{2r-1} \quad + \quad \frac{2r^4+6r^3+\frac{287}{2}r^2-\frac{303}{2}r}{9^2}N^{2r-2} \quad + \quad \text{lower order terms.}$$

Proof of Theorem 2. Define $G_N := (x-1)^N F_N(x)$. Then G_N has a Taylor series expansion about $x = 1$, whose first N coefficients are the Rademacher coefficients:

$$G_N = \sum_{j=0}^{N-1} C_{0,1,N-j}(N)(x-1)^j + O((x-1)^N).$$

Clearly,

$$(1-x^N)G_N = (x-1)G_{N-1}. \tag{3.5}$$

Expanding $(1-x^N)$ on the left hand side of (3.5) as a Taylor polynomial about $x = 1$, we have

$$\begin{aligned}\left(-\sum_{j=1}^{N}\binom{N}{j}(x-1)^j\right)\left(\sum_{j=0}^{N-1} C_{h,k,N-j}(N)(x-1)^j + \text{ higher degree terms}\right) \\ = \sum_{j=1}^{N-1} C_{0,1,N-j}(N-1)(x-1)^j + \text{higher degree terms}\end{aligned} \tag{3.6}$$

Comparing the coefficients of $(x-1)^1$ on both sides of (3.6), we find

$$-NC_{0,1,N}(N) = C_{0,1,N-1}(N-1).$$

Solving the recurrence with the initial condition $C_{0,1,1}(1) = -1$, yields

$$C_{0,1,N}(N) = \frac{(-1)^N}{N!}, \tag{3.7}$$

which is (3.1).

Comparing the coefficients of $(x-1)^2$ on both sides of (3.6), we find, taking into account (3.7),

$$-NC_{0,1,N-1}(N) - \binom{N}{2}\frac{(-1)^N}{N!} = C_{0,1,N-2}(N-1) \tag{3.8}$$

with initial condition $C_{0,1,1}(2) = -\frac{1}{4}$ yields

$$C_{0,1,N-1}(N) = \frac{(-1)^{N+1}}{4(N-2)!}, \tag{3.9}$$

which is (3.2).

Comparing the coefficients of $(x-1)^3$ on both sides of (3.6), we find, taking into account (3.7) and (3.9),

$$-NC_{0,1,N-2}(N) - \binom{N}{2}\frac{(-1)^{N+1}}{4(N-2)!} - \binom{N}{3}\frac{(-1)^N}{N!} = C_{0,1,N-3}(N-1) \tag{3.10}$$

with initial condition $C_{0,1,1}(3) = -\frac{17}{72}$ yields

$$C_{0,1,N-2}(N) = \frac{(-1)^N(9N^2 - 13N + 26)}{288(N-2)!}, \tag{3.11}$$

which is (3.3). Results for larger r follow analogously. □

3.2 $C_{1,2,l}(N)$

Let us now define $\bar{G}_N$, analogous to G_N. Let

$$\bar{G}_N := (x+1)^{\lfloor N/2 \rfloor} F_N(x).$$

Then $\bar{G}_N$ has a Taylor series expansion about $x = -1$, whose first $\lfloor \frac{N}{2} \rfloor$ coefficients are the Rademacher coefficients:

$$\bar{G}_N = \sum_{r=0}^{\lfloor N/2 \rfloor - 1} C_{1,2,\lfloor N/2 \rfloor - r}(N)(x+1)^r + O((x+1)^{\lfloor N/2 \rfloor}).$$

We now abandon the use of the floor function. Notice that

$$(1 - x^{2n-1})(1 - x^{2n})\bar{G}_{2n} = (x+1)\bar{G}_{2n-2}. \tag{3.12}$$

Thus, by expanding the two left most factors on the left side as a Taylor series about $x = -1$,

$$\begin{aligned} &\left\{ \sum_{r=1}^{4n-1} (-1)^{r+1} \left[\binom{4n-1}{r} + \binom{2n}{r} - \binom{2n-1}{r} \right] (x+1)^r \right\} \\ &\qquad \times \left(\sum_{r=0}^{n-1} C_{1,2,n-r}(2n)(x+1)^r + \text{ higher degree terms} \right) \\ &\qquad\qquad = \sum_{r=1}^{n-1} C_{1,2,n-r}(2n-2)(x+1)^r + \text{ higher degree terms} \end{aligned} \tag{3.13}$$

By comparing coefficients of $(x+1)^r$ in both sides of (3.13) and solving the recurrences, we obtain formulas for $C_{1,2,n-r}(2n)$ analogous to those for $C_{0,1,N-r}(N)$.

Comparing coefficients of $(x+1)$ on both sides of (3.13), we find

$$4nC_{1,2,n}(2n) = C_{1,2,n-1}(2n-2),$$

which, together with initial condition $C_{1,2,1}(2) = \frac{1}{4}$, yields

$$C_{1,2,n}(2n) = \frac{1}{2^{2n} n!}. \tag{3.14}$$

Comparing coefficients of $(x+1)^2$ on both sides of (3.13), and taking into account (3.14), we find

$$4nC_{1,2,n-1}(2n) - \frac{2n-1}{4^{n-1}(n-1)!} = C_{1,2,n-2}(2n-2),$$

which, together with initial condition $C_{1,2,1}(4) = \frac{1}{8}$, yields

$$C_{1,2,n-1}(2n) = \frac{n}{2^{2n}(n-1)!}. \tag{3.15}$$

Comparing coefficients of $(x+1)^3$ on both sides of (3.13), and taking into account (3.14) and (3.15), we find

$$4nC_{1,2,n-2}(2n) - \frac{4n^2(2n-1)}{4^n(n-1)!} + \frac{n(2n-1)(16n-13)}{3\cdot 4^n n!} = C_{1,2,n-3}(2n-2),$$

which, together with initial condition $C_{1,2,1}(6) = \frac{461}{4608}$, yields

$$C_{1,2,n-2}(2n) = \frac{18n^3 - 8n^2 + 15n + 2}{3^2 \cdot 2^{2n+2}(n-1)!}. \tag{3.16}$$

This process can be continued indefinitely with higher powers of $(x+1)$, e.g.

$$C_{1,2,n-3}(2n) = \frac{(n^2+1)(6n^3 - 8n^2 + 9n + 2)}{3^2 \cdot 2^{2n+2}(n-1)!}, \tag{3.17}$$

$$C_{1,2,n-4}(2n) = \frac{2700n^7 - 7200n^6 + 15100n^5 - 17448n^4 + 21205n^3 - 8880n^2 + 13350n + 3423}{25 \cdot 3^4 \cdot 2^{2n+5}(n-1)!}. \tag{3.18}$$

Of course, the observation

$$(1-x^{2n})(1-x^{2n+1})\bar{G}_{2n+1} = (x+1)\bar{G}_{2n-1}$$

leads to analogous formulas for the $C_{1,2,n-r}(2n+1)$, e.g.,

$$C_{1,2,n}(2n+1) = \frac{1}{2^{2n+1}\ n!}, \tag{3.19}$$

$$C_{1,2,n-1}(2n+1) = \frac{2n^2+2n+1}{2^{2n+2}\ n!}, \tag{3.20}$$

$$C_{1,2,n-2}(2n+1) = \frac{18n^5 + 46n^4 + 61n^3 + 53n^2 + 29n + 9}{9 \cdot 2^{2n+3}\ (n+1)!}, \tag{3.21}$$

$$C_{1,2,n-3}(2n+1) = \frac{(2n^2+2n+3)(6n^4 + 4n^3 + 3n^2 + 2n + 3)}{9 \cdot 2^{2n+4} n!}. \tag{3.22}$$

Clearly, the same idea can be used to find formulas for $C_{h,k,n-j}(kn+r)$ for any h, k, j, r. This has been implemented in the procedure `ChkFormula` in the `HANS` Maple package. For those desiring automatically generated papers, containing both formulas of this type and their proofs, please use the `HansTopDownAutoPaper` procedure in the `HANS` Maple package.

4 Close Encounters of the Rademacher Kind

While it appears that $\lim_{N\to\infty} C_{h,k,l}(N)$ does not exist for any (h,k,l), we can nonetheless define $B_{h,k,l}(N)$ to be the N which comes closest to Rademacher's conjectured value of $C_{h,k,l}(\infty)$. This is implemented in the `CloseEncounters` procedure in `HANS`.

l	$B_{0,1,l}(N)$	absolute difference	absolute ratio
1	25	0.0003177	0.99989
2	47	0.0001434	0.99924
3	71	0.0000828	0.99991
4	149	0.0000009	1.00001
5	109		

Notice that the first few values of $B_{0,1,l}(N)$ are close to $24l$. This motivates us to consider comparing $C_{0,1,l}(24l)$ to $R_{0,1,l}$, Rademacher's conjectured value of $C_{0,1,l}(\infty)$.

l	$\lvert C_{0,1,l}(24l) - R_{0,1,l}\rvert$	$\lvert C_{0,1,l}(24l)/R_{0,1,l}\rvert$
1	0.0053741095	1.018346206
2	0.0015044594	1.007927400
3	0.00033240887	0.996241370
4	0.00004427030	1.001376635
5	0.000011288321	0.9988220859
6	0.000001686611	1.0006971253
7	0.0000001275687	0.9997575030
8	0.0000000110523	1.0000986383
9	0.00000000239242	0.9999562770
10	0.000000005333208	1.0000141594
11	0.0000000187490584	0.9999947242
12	0.0000000393434274	1.0000017401

Thus we have some evidence that even though the N for which $C_{0,1,l}(N)$ is closest to $R_{0,1,l}$ is not $N = 24l$, $C_{0,1,l}(24l)$ seems to provide a good approximation to $R_{0,1,l}$.

References

[1] G. E. Andrews, Partitions: at the interface of q-series and modular forms, *Ramanujan J.* **7** (2003) 385–400.

[2] L. Ehrenpreis, Function theory for Rogers-Ramanujan-like partition identities, in *A Tribute to Emil Grosswald: Number Theory and Related Analysis*, ed. M. I. Knopp and M. Sheingorn, *Contemporary Math* **143** (1993) 259–320.

[3] A. O. Munagi, The Rademacher conjecture and q-partial fractions, *Ramanujan J.* **15** (2008) 339–347.

[4] H. Rademacher, *Topics in Analytic Number Theory*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band 169, Springer, 1973.