**An Umbral Approach to the Hankel Transform for Sequences**

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Recall that the *binomial transform* of a sequence $\{a_n\}_{n=0}^{\infty}$, let's call it $BIN(a) = \{b_n\}_{n=0}^{\infty}$, is defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad ,$$

and the *Hankel transform* ([L][SpSt]), let's call it $H(a) = \{h_n\}_{n=0}^{\infty}$, is defined by

$$h_n = \det(a_{i+j})_{0 \le i,j \le n} \quad .$$

In [L] (see also [SpSt]) it was proved that $H \circ BIN = H$, in other words, for any sequence $\{a_n\}_{n=0}^{\infty}$, $H(BIN(a)) = H(a)$. Let's prove it à la Gian-Carlo Rota, using *umbras*. For any sequence $\{c_n\}_{n=0}^{\infty}$ associate an *umbra* (linear functional on K[z], K any commutative ring), defined by $C(z^i) := c_i (0 \le i < \infty)$, and **extended linearly**. Let $A$ and $B$ be the umbras associated with $\{a_n\}_{n=0}^{\infty}$, and $b := BIN(a)$ respectively. Then

$$B(z^n) = b_n = \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{k=0}^{n} \binom{n}{k} A(z^k) = A\left( \sum_{k=0}^{n} \binom{n}{k} z^k \right) = A\left((1+z)^n\right) \quad ,$$

for every $n \ge 0$, and hence by **linearity**

$$B(p(z)) = A(p(z+1)) \quad , \quad for\ every\ p(z) \in K[z] \quad .$$

We can also express the Hankel sequence transform umbrally. If $\{h_n\}_{n=0}^{\infty}$ is the Hankel transform of $\{a_n\}_{n=0}^{\infty}$ then

$$h_n = \det(a_{i+j})_{0 \le i,j \le n} = \det(A_{x_j}(x_j^{i+j}))_{0 \le i,j \le n} = A_{x_0} A_{x_1} \dots A_{x_n} \left( \det(x_j^{i+j})_{0 \le i,j \le n} \right) =$$

$$A_{x_0} A_{x_1} \dots A_{x_n} \left( x_0^0 x_1^1 \cdots x_n^n \prod_{0 \le i < j \le n} (x_i - x_j) \right) = \frac{1}{n!} A_{x_0} A_{x_1} \dots A_{x_n} \left( \prod_{0 \le i < j \le n} (x_i - x_j)^2 \right) \quad ,$$

using multi-linearity, the Vandermonde determinant, and symmetrizing (plus Vandermonde again), respectively. Equating left and right, we have:

$$H(a)_n = \frac{1}{n!} A_{x_0} A_{x_1} \dots A_{x_n} \left( \prod_{0 \le i < j \le n} (x_i - x_j)^2 \right) \quad .$$

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This immediately implies that the Hankel transform is preserved under the binomial transform, since the discriminant is invariant under $x_i \to 1 + x_i$.

**Remarks. 1.** Similarly one can prove umbrally the various analogs in [L] and [SpSt]. **2.** One can deduce explicit Hankel evaluations for $a_n = n!$ (and hence for any sequence (like the derangement numbers,) binomially related to it), by using $A(z^n) = \int_0^\infty e^{-z} z^n \, dz$, and invoking specializations of Mehta's integral. Ditto for Macmahon and Carlitz determinants, and their generalizations, using Selberg's integral. **3.** One can view the Hankel transform without using determinants explicitly, just as a certain multi-linear form. It may be fun to talk about analogs using other 'semi-invariants' (in the sense of Cayley and Sylvester), as well as analogs of the discriminant for other *root systems* (the square of the Weyl denominator), and then use Macdonald integrals.

**References**

[L] John W. Layman, *The Hankel transform and some of its properties*, Journal of Integer Sequences **4:01.1.5**, 2001.

[SpSt] Michael Z. Spivey and Laura L. Steil, *The k-binomial transforms and the Hankel transform*, preprint. See also the abstract to Spivey's talk at the Integers 2005 Ron Graham Conference.