On An Intriguing Property of the Center of Mass of Points on a Sphere in R^d

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Abstract: We present a short proof of an intriguing result proved in 2007 by Leonid Hanin, Robert Fisher, and Boris Hanin.

In a delightful article [HFH], the authors first stated, and gave a beautiful *synthetic* proof, of the case $d = 2$ of the following two theorems regarding points on a circle. They later considered general quadratic surfaces, focusing on two and three dimensions, and remarked that their reasoning is true for any dimension.

Theorem 1: Let $X_1, \ldots, X_n$ be $n \geq 2$ points on a sphere in R^d, and C be their geometric center of mass. Denote by $Y_1, \ldots, Y_n$ the second points of intersection of the lines $X_1C, X_2C, \ldots, X_nC$ with the sphere, respectively, then

$$\sum_{i=1}^{n} \frac{X_iC}{CY_i} = n \quad .$$

Theorem 2: Let $X_1, \ldots, X_n$ be $n \geq 2$ points on a sphere in R^d, with center O, and let C be their geometric center of mass. For any point P inside the sphere, let $Y_1, \ldots, Y_n$ be the second points of intersection of PX_i with the sphere. The set of points P for which

$$\sum_{i=1}^{n} \frac{X_iP}{PY_i} = n \quad ,$$

is a sphere with diameter OC.

In this note I will give a short, self-contained, proof of their result for general d, but for the sake of simplicity will stick to spheres. Of course, by a change of variables every quadratic surface can be transformed to a sphere, if you don't mind "virtual points".

Since Theorem 2 implies Theorem 1 we will only prove the former. We need the following simple lemma.

Lemma: For any point X on the unit d-dimensional sphere, and any point P inside the sphere , let Y be the second point of intersection of the line XP with the sphere, then

$$\frac{XP}{PY} = \frac{2(X,P) - (P,P) - 1}{(P,P) - 1} \quad .$$

Proof of the Lemma: Every point on the line joining X and P has the form $X + s(P - X) = (1 - s)X + sP$ where $s = 0$, corresponds to X, and $s = 1$ corresponds to P. It meets the sphere when s satisfies

$$((1 - s)X + sP, (1 - s)X + sP) - 1 = 0 \quad .$$

Expanding, we get

$$(1-s)^2 \cdot (X,X) + 2(1-s)s \cdot (X,P) + s^2 \cdot (P,P) - 1 = 0 \quad .$$

Using $(X,X)=1$ we get

$$s(-2+s+2(1-s)\cdot(X,P)+s\cdot(P,P)) = 0 \quad .$$

This equation has two solutions: $s=0$ corresponds to X, and the one corresponding to Y is:

$$s = \frac{2(1-(X,P))}{1-2(X,P)+(P,P)} \quad .$$

Hence

$$\frac{XP}{PY} = \frac{1}{s-1} = \frac{2(X,P)-(P,P)-1}{(P,P)-1} \qquad . \quad \square$$

Proof of Theorem 2: Without loss of generality, the sphere is centered at the origin, and has radius 1. The center of mass of the points X_i is

$$C := \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) \quad .$$

The condition

$$\sum_{i=1}^{n} \frac{X_i P}{P Y_i} \, = \, n \quad ,$$

thanks to the lemma, is

$$\sum_{i=1}^{n} \frac{2(X_i,P)-(P,P)-1}{(P,P)-1} = \, n \quad ,$$

which is easily seen to be equivalent to

$$(P - C/2, P - C/2) = (C/2, C/2) \quad . \quad \square$$

Reference

[HFH] Leonid G. Hanin, Robert J. Fisher, and Boris L. Hanin, *An intriguing property of the center of mass for points on quadratic curves and surfaces*, Mathematics Magazine **80**, No. 5, (December 2007), 353-362.

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