Exponents

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Algebra

In algebra we have **symbols** and **numbers** mixed up. The notation is confusing. ab means a times b, i.e. $a \times b$, but we don't write the $\times$. But for numbers, we have to write the $\times$, since 23 means "twenty-three" and not 2 times 3.

In algebra we have **parantheses**, which means *do it first.* They are very important.

Example: $3(a+b)$, means: first add up a and b then multiply by 3. They are rules for *getting rid* of parentheses, called **foiling**. In this case we have $3(a)+3(b)=3a+3b$.

When we pronounce $3(a+b)$ we say "3 times the quantity $a+b$.

Precedence: Mathematicians are lazy and often they don't write parentheses explicity, but use **conventions** of precedence.

First: All powers

Second: All multiplications and divisions

Third: all additions and subtractions

So $a+bc+d$ is really shorthand for $a+(bc)+d$. This is not to be confused with $(a+b)(c+d)$ which means something else completely.

Problem: Do $(3+11)(8-4)+7\cdot 2-8/2$.

Exponents:

a^n means "a times a times a ... times a", where there are n of the a's.

You can always **spell-out** numerical or symbolic powers.

Examples: $2^3=2\times 2\times 2=8$.

$a^4=aaaa$.

Important convention:

$$a^{-n}=\frac{1}{a^n}$$

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Examples:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$
$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$
$$(1/4)^{-2} = \frac{1}{(1/4)^2} = \frac{1}{1/16} = 16$$

Reminder: "The bottom of the bottom goes to the top"

$$\frac{1}{\frac{1}{Whatever}} = Whatever \quad .$$

The Spelling-Out Method

$$\frac{3^{-2}}{2^{-2}} = \frac{1/9}{1/4} = 4/9 \quad .$$

Watch Out from a Common Mistake: The following is wrong!!!!

$$\frac{3^{-2}}{2^{-2}} = \frac{3}{2} \quad .$$

You can't "cancel-out" the the same thing (in our case -2) when they are exponents.

To be safe, always do the numerical part of exponent expressions by the spelling-out method.

Examples:

$5^2 = 5 \cdot 5 = 25$,

$(-5)^2 = (-5) \cdot (-5) = 25$ (remember: minus times minus is plus),

$-5^2 = -(5^2) = -25$.

Watch out: -5^2 is **not** $(-5)^2$. By the precedence rules you **first** do the power, $5^2 = 25$ and only **then** you do the $-$, getting -25. -5^2 is the same as $-(5^2)$. On the other-hand, if they want you to do the $-$ first and then do the power, then they need parantheses $(-5)^2$.

Do right now:

$2^3 =$

$(-2)^4 =$

$-2^4 =$

$-(-5)^2 =$

$-0.2^3 =$

$(-0.2)^3 =$

$(-\frac{3}{5})^2 =$

Properties of Exponents

Rule 1:

$$(ab)^n = a^n b^n \quad .$$

Why is it true?: For example, when $n = 3$, we have

$$(ab)^3 = (ab)(ab)(ab) = ababab = aaabbb = a^3b^3 \quad .$$

Rule 2:

$$x^n x^m = x^{n+m} \quad .$$

Why is it true?: For example, when $n = 3, m = 4$, we have

$$x^3x^4 = (xxx)(xxxx) = xxxxxxx = x^7 \quad .$$

Rule 3:

$$\frac{x^n}{x^m} = x^{n-m} \quad .$$

Why is it true?: For example, when $n = 5, m = 3$, we have

$$\frac{x^5}{x^3} = \frac{xxxxx}{xxx} = xx = x^2 \quad .$$

Rule 4

$$(x^m)^n = x^{mn} \quad .$$

Why is it true?: For example, when $n = 3, m = 4$, we have

$$(x^3)^4 = (x^3)(x^3)(x^3)(x^3) = (xxx)(xxx)(xxx)(xxx) = xxxxxxxxxxxx = x^{12} \quad .$$

Examples:

$x^5 \cdot x^4 = x^{5+4} = x^9$

$z^2 \cdot z^5 = z^{2+5} = z^7$

$(3^2)^5 = 3^{2\cdot 5} = 3^{10}$

$$\left(-\frac{2}{3}x^2\right)^3 = (-1)^3 \frac{2^3}{3^3}(x^2)^3 = -\frac{8}{27}x^6 \quad .$$

$$5a^7(-4a^6) = (5)(-4)a^7 \cdot a^6 = -20a^{7+6} = -20a^{13} \quad .$$

Common Mistake (Watch Out!):

$$(a^6)(a^7) = a^{42}(WRONG!!!)$$

$(a^6)^7 = a^{42}$ is right!, but if 6 and 7 are at the **same level** then $(a^6)(a^7) = a^{6+7} = a^{13}$, i.e. you **add** the exponents **not** multiply them. But in

$$(a^6)^7$$

the 7 is “above” the 6, so you **multiply** .

Do right now!

$-3a^2(4a^4) =$

$-3b^3a^2(3a^5b)(2ab^2) =$

$(3^3)^4 =$

$(2^5)^2 \cdot (2^2)^5 \cdot 2^4 =$

Writing with positive exponents

Remember

$$x^{-n} = \frac{1}{x^n} \quad .$$

Examples:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$(-5)^{-2} = \frac{1}{(-5)^2} = \frac{1}{25}$$

$$(-2)^{-5} = \frac{1}{(-2)^5} = \frac{1}{-32} = -\frac{1}{32} \quad .$$

$$(\frac{3}{4})^{-2} = \frac{3^{-2}}{4^{-2}} = \frac{4^2}{3^2} = \frac{16}{9} \quad .$$

Do Right Now!:

$$\left(\frac{2}{3}\right)^{-3} =$$

$$\left(\frac{1}{3}\right)^{-2} + \left(\frac{1}{2}\right)^{-3} =$$

$$\left(\frac{1}{4}\right)^{-3} + \left(\frac{1}{4}\right)^{-2} =$$

Simplifying complicated expressions

Example

$$(5y^4)^{-3}(2y^{-2})^3 = (5^{-3}2^3)y^{-12} \cdot y^{-6} = \frac{8}{125}y^{-18} = \frac{8}{125y^{18}} \quad .$$

Do right now!

$$(3z^{-2})^3(4z^{-3})^2 =$$

$$(5w^{-4})^2(4w^{-5})^3 =$$

Dividing Exponents

Remember:

$$\frac{x^m}{x^n} = x^{m-n}$$

Careful: Don't do things in your head! Do it step-by-step. It is very easy to make mistakes by doing "mental math", remember "minus minus is plus".

Examples:

$$\frac{x^{-1}}{x^9} = x^{-1-9} = x^{-10} = \frac{1}{x^{10}} \quad .$$

$$\frac{t^{-10}}{t^{-4}} = t^{-10--4} = t^{-6} = \frac{1}{t^6} \quad .$$

$$\left(\frac{8x^2y}{4x^4y^{-3}}\right)^4 = \left(\frac{2x^2y}{x^4y^{-3}}\right)^4 =$$

$$2^4(x^{-2}y^4)^4 = 16(x^{-8}y^{16}) = \frac{16y^{16}}{x^8} \quad .$$

Do Right Now!

$$\frac{z^{-10}}{z^{-5}} =$$

$$\frac{w^{-3}}{w^8} =$$

$$\frac{x^{13}}{x^{18}} =$$

$$\left(\frac{6x^3y^2}{3xy^{-2}}\right)^2 =$$

$$\left(\frac{16x^{-3}y^{-2}z^3}{4x^{-1}y^2z^2}\right)^4 =$$

$$\left(\frac{9x^3y^3z^3}{3x^{-1}y^{-2}z^{-2}}\right)^{-1} =$$