**An Inelegant (but Short(!)) Proof of a Major Index Theorem of Garsia and Gessel (Verbose Version)**

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Recall Adriano Garsia's favorite notation: $\chi(A) = 1$ if $A$ is true, and $\chi(A) = 0$ if $A$ is false. For example $\chi(MathIsFun) = 1$, $\chi(HeleneBarceloIsAGoodEditor) = 0$. Also recall that the *major index* of any list of integers $\pi = (\pi_1, \dots, \pi_n)$ (in particular a permutation) is defined by $maj(\pi) := \sum_{i=1}^{n-1} i\chi(\pi_i > \pi_{i+1})$. Let $\theta = \theta_1 \dots \theta_a$ and $\pi = \pi_1 \dots \pi_b$ be two disjoint lists of distinct integers. Let $S(\theta, \pi)$ be the set of (a+b)!/(a!b!) mergings (suffles) of $\theta$ and $\pi$. Let $(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$. Moti Novick[N] has recently found an elegant bijective proof of the following

**Theorem** (Garsia and Gessel[GG]):

$$\sum_{\sigma \in S(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b}}{(q)_a (q)_b} q^{maj(\theta)+maj(\pi)} \quad . \qquad (Moti)$$

In this short note I will present an *inelegant*, induction proof, by proving, more generally:

**Lemma**: Let $S_1(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is $\theta_a$, and let $S_2(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is $\pi_b$, then

$$\sum_{\sigma \in S_1(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_{a-1} (q)_b} q^{maj(\theta)+maj(\pi)+b\chi(\pi_b > \theta_a)} \quad , \qquad (Adriano)$$

$$\sum_{\sigma \in S_2(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_a (q)_{b-1}} q^{maj(\theta)+maj(\pi)+a\chi(\pi_b < \theta_a)} \quad . \qquad (Ira)$$

Note that adding-up ($Adriano$) and ($Ira$) gives ($Moti$). Let's call the left-sides of ($Adriano$) and ($Ira$) $F_1(a,b)$ and $F_2(a,b)$ respectively, and let's call their right-sides $G_1(a,b)$ and $G_2(a,b)$ respectively. It is immediate that $(F_1, F_2) = (G_1, G_2)$ when $a = 0$ or $b = 0$, and the fact that $(F_1, F_2) = (G_1, G_2)$ for all $a, b \geq 0$ follows from the fact that both $(X_1, X_2) = (F_1, F_2)$ and $(X_1, X_2) = (G_1, G_2)$ satisfy the recurrence

$$X_1(a,b) = q^{(a+b-1)\chi(\theta_{a-1} > \theta_a)} X_1(a-1,b) + q^{(a+b-1)\chi(\pi_b > \theta_a)} X_2(a-1,b)$$

$$X_2(a,b) = q^{(a+b-1)\chi(\theta_a > \pi_b)} X_1(a,b-1) + q^{(a+b-1)\chi(\pi_{b-1} > \pi_b)} X_2(a,b-1) \quad .$$

The proof for $(F_1, F_2)$ is left to the reader. Let's prove it for $(G_1, G_2)$. We will only prove the first identity. The second one is also left to the reader.

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We have to prove that

$$G_1(a,b) - q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)}G_1(a-1,b) - q^{(a+b-1)\chi(\pi_b>\theta_a)}G_2(a-1,b) = 0 \quad . \qquad (HaImEfes)$$

By definition:

$$G_1(a,b) = \frac{(q)_{a+b-1}}{(q)_{a-1}(q)_b} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)} \quad ,$$

$$G_2(a,b) = \frac{(q)_{a+b-1}}{(q)_a(q)_{b-1}} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+a\chi(\pi_b<\theta_a)} \quad .$$

Replacing $a$ by $a-1$ we have

$$G_1(a-1,b) = \frac{(q)_{a+b-2}}{(q)_{a-2}(q)_b} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})} \quad ,$$

$$G_2(a-1,b) = \frac{(q)_{a+b-2}}{(q)_{a-1}(q)_{b-1}} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} \quad .$$

Substituting in the left side of $(HaImEfes)$, we have to prove:

$$\frac{(q)_{a+b-1}}{(q)_{a-1}(q)_b} q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)}$$

$$-q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot \frac{(q)_{a+b-2}}{(q)_{a-2}(q)_b} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot \frac{(q)_{a+b-2}}{(q)_{a-1}(q)_{b-1}} q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} = 0$$

Dividing by $\frac{(q)_{a+b-2}}{(q)_{a-1}(q)_b}$, we have to prove

$$(1-q^{a+b-1})q^{maj(\theta_1\ldots\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)}$$

$$-q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1})q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot (1-q^b)q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+(a-1)\chi(\pi_b<\theta_{a-1})} = 0 \quad .$$

But

$$maj(\theta_1\ldots\theta_a) = maj(\theta_1\ldots\theta_{a-1}) + (a-1)\chi(\theta_{a-1}>\theta_a) \quad ,$$

so we have to prove:

$$(1-q^{a+b-1})q^{maj(\theta_1\ldots\theta_{a-1})+(a-1)\chi(\theta_{a-1}>\theta_a)+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_a)} -$$

$$q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1})q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)} \cdot (1-q^b)q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)+a\chi(\pi_b<\theta_{a-1})} = 0$$

Dividing both sides by $q^{maj(\theta_1\ldots\theta_{a-1})+maj(\pi_1\ldots\pi_b)}$, we have to prove

$$(1-q^{a+b-1})q^{(a-1)\chi(\theta_{a-1}>\theta_a)+b\chi(\pi_b>\theta_a)} - q^{(a+b-1)\chi(\theta_{a-1}>\theta_a)} \cdot (1-q^{a-1})q^{b\chi(\pi_b>\theta_{a-1})}$$

$$-q^{(a+b-1)\chi(\pi_b>\theta_a)}\cdot(1-q^b)q^{(a-1)\chi(\pi_b<\theta_{a-1})}=0$$

Combining powers of $q$, we have to prove:

$$(1-q^{a+b-1})q^{(a-1)\chi(\theta_{a-1}>\theta_a)+b\chi(\pi_b>\theta_a)}-(1-q^{a-1})q^{b\chi(\pi_b>\theta_{a-1})+(a+b-1)\chi(\theta_{a-1}>\theta_a)}$$

$$-(1-q^b)q^{(a-1)\chi(\pi_b<\theta_{a-1})+(a+b-1)\chi(\pi_b>\theta_a)}=0$$

We have **six** cases to consider according to the relative rankings of $\theta_{a-1},\theta_a,\pi_b$.

**Case 123**: $\theta_{a-1}<\theta_a<\pi_b$. We have to prove

$$(1-q^{a+b-1})q^b-(1-q^{a-1})q^b-(1-q^b)q^{(a+b-1)}=0 \quad .$$

Dividing by $q^b$, we have to prove

$$(1-q^{a+b-1})-(1-q^{a-1})-(1-q^b)q^{(a-1)}=0 \quad .$$

Expanding, we have, to prove

$$1-q^{a+b-1}-1+q^{a-1}-q^{a-1}+q^{a+b-1}=0 \quad ,$$

and this is indeed true.

**Case 132**: $\theta_{a-1}<\pi_b<\theta_a$. We have to prove

$$(1-q^{a+b-1})q^0-(1-q^{a-1})q^b-(1-q^b)q^0=0$$

Expanding, we get:

$$1-q^{a+b-1}-q^b+q^{a+b-1}-1+q^b=0 \quad ,$$

and this is indeed true.

**Case 213**: $\theta_a<\theta_{a-1}<\pi_b$:

$$(1-q^{a+b-1})q^{(a-1)+b}-(1-q^{a-1})q^{b+(a+b-1)}-(1-q^b)q^{a+b-1}=0 \quad ,$$

Dividing by $q^{a+b-1}$, we have to prove

$$(1-q^{a+b-1})-(1-q^{a-1})q^b-(1-q^b)=0 \quad ,$$

Expanding, we have to prove

$$1-q^{a+b-1}-q^b+q^{a+b-1}-1+q^b=0 \quad ,$$

and this is indeed true.

**Case 231**: $\theta_a < \pi_b < \theta_{a-1}$. We have to prove

$$(1-q^{a+b-1})q^{(a-1)+b} - (1-q^{a-1})q^{a+b-1} - (1-q^b)q^{a-1+(a+b-1)} = 0 \quad .$$

Dividing by $q^{a+b-1}$, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1}) - (1-q^b)q^{a-1} = 0 \quad .$$

Expanding, we have to prove

$$1 - q^{a+b-1} - 1 + q^{a-1} - q^{a-1} + q^{a+b-1} = 0 \quad ,$$

and this is indeed true.

**Case 312**: $\pi_b < \theta_{a-1} < \theta_a$:

$$(1-q^{a+b-1})q^0 - (1-q^{a-1})q^0 - (1-q^b)q^{a-1} = 0 \quad ,$$

Expanding, we have to prove

$$1 - q^{a+b-1} - 1 + q^{a-1} - q^{a-1} + q^{a+b-1} = 0 \quad ,$$

and this is indeed true.

**Case 321**: $\pi_b < \theta_a < \theta_{a-1}$:

$$(1-q^{a+b-1})q^{a-1} - (1-q^{a-1})q^{a+b-1} - (1-q^b)q^{a-1} = 0 \quad .$$

Dividing by $q^{a-1}$, we have to prove

$$(1-q^{a+b-1}) - (1-q^{a-1})q^b - (1-q^b) = 0 \quad .$$

Expanding, we have to prove

$$1 - q^{a+b-1} - q^b + q^{a+b-1} - 1 + q^b = 0 \quad ,$$

and this is indeed true. $\square$

**Remarks**: **1.** Moti Novick's elegant bijection also preserves equidistribution over Inverse Descent Classes. **2.** It would be interesting to bijectify the above inductive proof, along the lines of [MZ], and see if the resulting bijection is identitical, or similar, to Moti Novick's elegant bijection.

**References**:

[GG] Adriano M. Garsia and Ira Gessel, *Permutation statistics and partitions*, Advances in Mathematics **31**(1979), 288-305.

[MZ] Philip Matchett Wood and Doron Zeilberger, *A Translation Method for Finding Combinatorial Bijections*, to appear in Annals of Combinatorics.
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[N] Moti Novick, *A Bijective proof of a major index theorem of Garsia and Gessel*, preprint, Hebrew University of Jerusalem.