The Number of 1...d-Avoiding Permutations of Length d+r for SYMBOLIC d but Numeric r

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Dedicated to Ira Martin GESSEL (b. April 9, 1951), on his millionth$_2$ birthday

Preface: How many permutations are there of length googol+30 avoiding an increasing subsequence of length googol?

This number is way too big for our physical universe, but the number of permutations of length googol+30 that *contain* at least one increasing subsequence of length googol is a certain integer that may be viewed in `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/gessel64.pdf`. Hence the number of permutations of length googol+30 *avoiding* an increasing subsequence of length googol is $(googol + 30)!$ *minus* the above small number.

Counting the "Bad Guys"

Recall that thanks to Robinson-Schensted ([Rob][Sc]), the number of permutations of length n that do **not** contain an increasing subsequence of length d is given by

$$G_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) < d}} f_\lambda^2 \quad ,$$

where λ denotes a typical *Young diagram*, and f_λ is the number of *Standard Young tableaux* whose *shape* is λ.

Hence the number of permutations of length n that **do** contain an increasing subsequence of length d is

$$B_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) \geq d}} f_\lambda^2 \quad .$$

Since the total number of permutations of length n is $n!$ ([B]), if we know how to find $B_d(n)$, we would know immediately $G_d(n) = n! - B_d(n)$, at least if we leave $n!$ alone as a factorial, rather than spell it out.

Recall that the *Hook Length formula* (see [Wiki]) tells you that if λ is a Young diagram then

$$f_\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)} \quad ,$$

where the product is over all the n cells of the Young diagram, and the *hook-lenght*, $h(c)$, of a cell $c = (i, j)$, is $(\lambda_i - i) + (\lambda'_j - j) + 1$, where λ' is the *conjugate* diagram, where the rows become columns and vice-versa.

Let r be a fixed integer, then for *symbolic* d, valid for $d \geq r-1$, any Young diagram with at least d rows, and with $d+r$ cells, can be written, for some Young diagram $\mu = (\mu_1, \dots, \mu_r)$, with $\leq r$ cells, (where we add zeros to the end if the number of parts of μ is less than r) as

$$\lambda = (1+\mu_1, \dots, 1+\mu_r, 1^{d-r+r'}) \quad ,$$

where $r' = r - |\mu|$. For such a shape λ, with *at least* d rows,

$$\prod_{c \in \lambda} h(c) = \left(\prod_{c \in \mu} h(c) \right) \cdot ((d+r'+\mu_1)(d+r'-1+\mu_2) \cdots (d+r'-r+1+\mu_r)) \cdot (d-r+r')! \quad .$$

Hence f_λ, that is $(d+r)!$ divided by the above, is a certain specific number times a certain polynomial in d. Since, for a specific, *numeric*, r, there are only *finitely* many Young diagrams with at most r cells, the computer can find all of them, compute the polynomial corresponding to each of them, square it, and add-up all these terms, getting an *explicit* **polynomial** expression, in the variable d, for $B_d(d+r)$, the number of permutations of length $d+r$ that *contain* an increasing subsequence of length d. As we said above, from this we can find $G_d(d+r) = (d+r)! - B_d(d+r)$, valid for *symbolic* $d \geq r-1$.

$\mathbf{B_d(d+r)}$ for r from 0 to 30

$$B_d(d) = 1 \quad ,$$

$$B_d(d+1) = d^2 + 1 \quad ,$$

$$B_d(d+2) = \frac{1}{2}\,d^4 + d^3 + \frac{1}{2}\,d^2 + d + 3 \quad ,$$

$$B_d(d+3) = \frac{1}{6}\,d^6 + d^5 + \frac{5}{3}\,d^4 + \frac{2}{3}\,d^3 + \frac{19}{6}\,d^2 + \frac{31}{3}\,d + 11 \quad ,$$

$$B_d(d+4) = \frac{1}{24}\,d^8 + \frac{1}{2}\,d^7 + \frac{25}{12}\,d^6 + \frac{19}{6}\,d^5 + \frac{29}{24}\,d^4 + 9\,d^3 + \frac{247}{6}\,d^2 + \frac{395}{6}\,d + 47 \quad ,$$

$$B_d(d+5) = \frac{1}{120}\,d^{10} + \frac{1}{6}\,d^9 + \frac{31}{24}\,d^8 + \frac{14}{3}\,d^7 + \frac{823}{120}\,d^6 + \frac{67}{30}\,d^5 + \frac{653}{24}\,d^4 + \frac{959}{6}\,d^3 + \frac{10459}{30}\,d^2 + \frac{3981}{10}\,d + 239 \quad .$$

For $B_d(d+r)$ for r from 6 up to 30, see
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64a` .

Sequences

The sequence $G_3(n)$ is the greatest *celeb* in the kingdom of combinatorial sequences [the subject of an entire book([St]) by Ira Gessel's illustrious *academic father*, Richard Stanley], the super-famous **A000108** in Neil Sloane's legendary database ([Sl]). $G_4(n)$, while not in the same league as the Catalan sequence, is still moderately famous, **A005802**. $G_5(n)$ is **A047889**, $G_6(n)$ is **A047890**, $G_7(n)$ is **A052399** , $G_8(n)$ is **A072131**, $G_9(n)$ is **A072132**, $G_{10}(n)$ is **A072133**, $G_{11}(n)$ is **A072167**, but $G_d(n)$ for $d \geq 12$ are absent (for a good reason, one must stop somewhere!). Also the *flattened version* of the *double-sequence*, $\{G_d(n)\}$, for $1 \leq d \leq n \leq 45$ is **A047887**. Using the polynomials $B_d(d+r)$, we computed the first $2d+1$ terms of $G_d(n)$ for $d \leq 30$. See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64b` .

But this method can only go up to $2d+1$ terms of the sequence $G_d(n)$, and of course, the first $d-1$ terms are trivial, namely $n!$. Can we find the first 100 terms (or whatever) for the sequences $G_d(n)$ for d up to 20, and beyond, **efficiently?**

Encore: Efficient Computer-Algebra Implementation of Ira Gessel's AMAZING Determinant Formula

Recall Ira Gessel's [G] famous expression for the generating function of $G_d(n)/n!^2$, *canonized* in the *bible* ([W], p. 996, Eq. (5)). Here it is:

$$\sum_{n \geq 0} \frac{G_d(n)}{n!^2} x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\ldots,d} \quad ,$$

in which $I_\nu(t)$ is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\,t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Can we use this to compute the first 100 terms of, say, $G_{20}(n)$?

While computing *numerical* determinants is very fast, computing *symbolic* ones is a different story. First, do not get scared by the "infinite" power series. If we are only interested in the first N terms of $G_d(n)$, then it is safe to truncate the series up to t^{2N}, and take the determinant of a $d \times d$ matrix with *polynomial entries.* If you use the *vanilla* determinant in a computer-algebra system such as Maple, it would be very inefficient, since the degree of the determinant is much larger than $2N$. But a little cleverness can make things more efficient. The Maple package `Gessel64`, available free of charge from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Gessel64` ,

accompanying this article, has a procedure `SeqIra(k,N)` that computes the first N terms of $G_k(n)$, using a division-free algorithm (see [Rot]) over an appropriate ring to compute the determinant in Gessel's famous formula.

```
SeqIra:=proc(k,N) local ira,t,i,j, R:

  R := table():

  R[`0`] := 0:

  R[`1`] := 1:

  R[`+`] := `+`:
```

```
  R[`-`] := `-`:

  R[`*`] := proc(p, q): return add(coeff(p*q, t, i)*t**i, i=0..2*N): end:

  R[`=`] := proc(p, q): return evalb(p = q): end:

  ira:=expand(LinearAlgebra[Generic][Determinant][R](Matrix([seq([seq(Iv(abs(i-j),t,2*N),
                                                                   j=1..k-1)],
                                                                i=1..k-1)]))):

  [seq(coeff(ira,t,2*i)*i!**2,i=1..N)]:

end:
```

In the above code, procedure `Iv(v,t,N)` computes the truncated modified Bessel function that shows up in Gessel's determinant, and it is short enough to reproduce here:

```
Iv:=proc(v,t,N) local j:  add(t**(2*j+v)/j!/(j+v)!,j=0..trunc((N-v)/2)+1):  end:
```
.

Using this procedure, the first-named author computed (in 4507 seconds) the first 100 terms of each of the sequences $G_d(n)$ for $3 \leq d \leq 20$, and could have gone much further.

See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64c` .

HAPPY 64th BIRTHDAY, IRA!

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