A Short Proof that the number of (a,b)-parking functions of length n is $a(a+bn)^{n-1}$

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Abstract: We give a very short proof of the fact that the number of (a,b)-parking functions of length n equals $a(a+bn)^{n-1}$. This was first proved in 2003 by Kung and Yan, via a very long and torturous route, as a corollary of a more general result.

Recall [KoW] that a parking function of length n is a list of positive integers $x = (x_1, \dots, x_n)$ whose non-decreasing rearrangement, $x'_1 \leq \dots \leq x'_n$ satisfies $x'_i \leq i, (1 \leq i \leq n)$. Their number is famously $(n+1)^{n-1}$.

More generally, for any non-negative integers, a and b, an (a,b)-parking function of length n is a list of positive integers $(x_1, \dots, x_n)$ whose non-decreasing rearrangement satisfies: $x'_i \leq a+b(i-1), (1 \leq i \leq n)$. Note that $(1,1)$-parking functions are the usual ones.

In [KuY], after 13 pages of heavy-going math, as a corollary of a much more general result (Cor. 5.5 there), the authors proved the following fact.

Fact: The number of (a,b)-parking functions of length n is $a(a+bn)^{n-1}$.

Short Proof: Let $\mathcal{P}(n,a,b)$ be the set of (a,b)-parking functions of length n, and let $p(n,a,b) := |\mathcal{P}(n,a,b)|$. For $0 \leq r \leq n$, the subset consisting of those with exactly r 1s is in bijection with $\mathcal{B}_{n,r} \times \mathcal{P}(n-r, a+br-1, b)$, where $\mathcal{B}_{n,r}$ is the set of r-element subsets of $\{1,\dots,n\}$. It is given by: $x \rightarrow (S,y)$, where S is the subset of locations of x where the 1s reside, and y is obtained from x by deleting the 1s, and then subtracting one from every remaining entry, and since $x'_{r+1} - 1 \leq a + b\,r - 1$, the claim follows. Hence:

$$p(n,a,b) = \sum_{r=0}^{n} \binom{n}{r} p(n-r, a+br-1, b) \quad ,$$

that uniquely determines $p(n,a,b)$, subject to the initial conditions $p(0,a,b) = 1$ and $p(n,0,b) = 0$.

But $q(n,a,b) := a(a+bn)^{n-1}$ satisfies the very same recurrence, where p is replaced by q, namely:

$$a(a+bn)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} (a+br-1)(a+bn-1)^{n-r-1} \quad ,$$

(Check![1]), and hence $p(n,a,b) = q(n,a,b)$ follows by induction on n. $\square$

[1] Write $a + b\,r - 1$ as $(a + b\,n - 1) - b\,(n-r)$, use $(n-r)\binom{n}{r} = n\binom{n-1}{r}$ and then invoke the binomial theorem twice.

Comment:

We were unaware of this result, and found it *ab initio*, by using *experimental mathematics*, via the Maple package

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/GenPark.txt` . We thank Lucy Martinez for telling us about [KuY].

Added Dec. 20, 2024: It turns out that this is **not** the first short proof. Richard Stanely emailed us with the following:

In connection with your recent arXiv paper with AJ Bu, you might be interested in the paragraph preceding Theorem 1.2 of the paper

Richard P. Stanley and Yinghui Wang, *Some aspects of* (r, k)*-parking functions*, J. Combinatorial Theory (A) **159** (2018), 54-78, `https://math.mit.edu/~rstan/papers/pf.pdf` .

It also turns out (see reference [7] in the above paper) that this result is much older than [KuY], it goes back to 1969 and is due to G. P. Steck.

References

[KoW] Alan G. Konheim and Benjamin Weiss, *An occupancy discipline and applications*, SIAM J. Applied Math. **14** (1966), 1266-1274. [Available from JSTOR.]

[KuY] Joseph P.S. Kung and Catherine Yan, *Gončarov polynomials and parking functions*, J. of Combinatorial Theory Ser. A, **102** (2003), 16-37.

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