A BINOMIAL COEFFICIENT IDENTITY ASSOCIATED TO A CONJECTURE OF BEUKERS

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If n is a positive integer, then let $A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$, and define integers $a(n)$ by $\sum_{n=1}^{\infty} a(n)q^n := q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - \cdots$. Beukers conjectured that if p is an odd prime, then

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}. \tag{1}$$

In [A-O] it is shown that (1) is implied by the truth of the following identity.

Theorem. *If n is a positive integer, then*

$$\sum_{k=1}^{n} k\binom{n}{k}^2\binom{n+k}{k}^2\left\{\frac{1}{2k} + \sum_{i=1}^{n+k}\frac{1}{i} + \sum_{i=1}^{n-k}\frac{1}{i} - 2\sum_{i=1}^{k}\frac{1}{i}\right\} = 0.$$

Remark. This identity is easily verified using the WZ method, in a generalized form [Z] that applies when the summand is a hypergeometric term times a WZ potential function. It holds for all positive n, since it holds for n=1,2,3 (check!), and since the sequence defined by the sum satisfies a certain (homog.) third order linear recurrence equation. To find the recurrence, and its proof, download the Maple package EKHAD and the Maple program zeilWZP from `http://www.math.temple.edu/~ zeilberg` . Calling the quantity inside the braces $c(n,k)$, compute the WZ pair (F,G), where $F = c(n,k+1) - c(n,k)$ and $G = c(n+1,k) - c(n,k)$. Go into Maple, and type `read zeilWZP; zeilWZP(k*(n+k)!**2/k!**4/(n-k)!**2,F,G,k,n,N):`

References

[A-O] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences* **(in preparation)**.

[B] F. Beukers, *Another congruence for Apéry numbers*, J. Number Th. **25** (1987), 201-210.

[Z] D. Zeilberger, *Closed Form (pun intended!)*, Contemporary Mathematics **143** (1993), 579-607.

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