A Fibonacci-Counting Proof Begged by Benjamin and Quinn

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Arthur Benjamin and Jennifer Quinn, in their delightful book[1], asked for *proofs that count* of the identities

$$\sum_i \binom{2n}{i} F_{2i} = 5^n F_{2n} \quad , \quad \sum_i \binom{2n}{i} F_{2i+1} = 5^n F_{2n+1} \quad .$$

Here goes. For any vector of integers u, let $|u|$ denote the sum of its entries. Let, for $\epsilon = 1, 0$,

$$A_\epsilon(n) := \{ (w,u) : w \in \{0,1\}^{2n} \quad , \quad u \in \{1,2\}^* \quad , \quad 2|w| - |u| = \epsilon \quad \} \quad ,$$

$$B_\epsilon(n) := \{ (w,u) : w \in \{1,2,3,4,5\}^{n} \quad , \quad u \in \{1,2\}^* \quad , \quad |u| = 2n - \epsilon \quad \} \quad .$$

The left sides of the identities *count* $A_\epsilon(n)$ and the right sides count $B_\epsilon(n)$, $(\epsilon = 1, 0)$. Consider a directed graph with vertices C_0 and C_1, where there are ten edges joining C_0 to C_0, five edges joining C_0 to C_1, five edges joining C_1 to C_0, and five edges joining C_1 to C_1. I claim that both $A_\epsilon(n)$ and $B_\epsilon(n)$ are in natural bijection with the set of n-step walks on this directed graph that start at C_ϵ $(\epsilon = 0, 1)$.

If $(w,u) \in A_1(n)$, start your journey at vertex C_1. Let u' be the shortest prefix of u such that $|u'| \geq 2(w_1 + w_2) - 1$, and write $u = u'u''$. Of course $|u'| = 2(w_1 + w_2) - 1$ or $|u'| = 2(w_1 + w_2)$. There are five possibilities for each of these scenarios, and once and for all, name the five edges from C_1 to C_0 and the five edges from C_1 to C_1 by these possibilities, respectively, and follow the appropriate edge, at the same time shrinking (w,u) by replacing it by (w', u'') where w' is w with its first two entries chopped. In the former case (w', u'') belongs to $A_0(n-1)$, and in the latter case to $A_1(n-1)$. Now continue recursively.

If $(w,u) \in A_0(n)$, start your journey at vertex C_0. Let u' be the shortest prefix of u such that $|u'| \geq 2(w_1 + w_2)$, and write $u = u'u''$. Of course $|u'| = 2(w_1 + w_2)$ or $|u'| = 2(w_1 + w_2) + 1$. There are ten possibilities for the former scenario and five possibilities for the latter. Name, once and for all, the ten edges from C_0 to C_0, and the five edges from C_0 to C_1 by these possibilities, respectively, and follow the appropriate edge, at the same time shrinking (w,u) by replacing it by (w', u'') where w' is w with its first two entries chopped. In the former case (w', u'') belongs to $A_0(n-1)$, and in the latter case to $A_1(n-1)$. Now continue recursively.

In the same vein, but much simpler, $B_0(n)$ and $B_1(n)$ correspond to n-step walks staring at C_0 and C_1 on that very same directed graph. The details are left to the reader.

REFERENCE

1. A. Benjamin and J. Quinn, "*Proofs that Really Count: The Art of Combinatorial Proofs*", The Math. Assoc. of Amer. 2003.

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