A Fibonacci-Counting Proof Begged by Benjamin and Quinn

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Arthur Benjamin and Jennifer Quinn, in their delightful book[1], asked for *proofs that count* of the case $\epsilon = 0$ of

$$\sum_i \binom{2n}{i} F_{2i+\epsilon} = 5^n F_{2n+\epsilon} \quad (\epsilon = 0, 1) \quad .$$

Here goes. For any vector of integers u, let $|u|$ denote the sum of its entries. Let, for $\epsilon = 1, 0$,

$$A_\epsilon(n) := \{(w, u) : w \in \{0,1\}^{2n} \quad , \quad u \in \{1,2\}^* \quad , \quad 2|w| - |u| = \epsilon \quad \} \quad ,$$

$$B_\epsilon(n) := \{(w, u) : w \in \{1,2,3,4,5\}^{n} \quad , \quad u \in \{1,2\}^* \quad , \quad |u| = 2n - \epsilon \quad \} \quad .$$

The left side *counts* $A_\epsilon(n)$ and the right side counts $B_\epsilon(n)$, $(\epsilon = 1, 0)$. Consider a directed graph with vertices A_0 and A_1, with ten edges from A_0 to A_0 and five edges each from A_0 to A_1, A_1 to A_0, A_1 to A_1. Every pair (w, u) of $A_\epsilon(n)$ corresponds to an n-step walk on this graph that starts at A_ϵ $(\epsilon = 0, 1)$, as follows.

Indeed, write $w = w'w''$, where $length(w') = 2$. For (w, u) of $A_\epsilon(n)$ $(\epsilon = 0, 1)$, let u'' be the longest suffix of u such that (w'', u'') belongs to either $A_0(n-1)$ or $A_1(n-1)$, and write $u = u'u''$. Equivalently, let u' be the shortest prefix such that (w', u') belongs to $A_0(1)$ or $A_1(1)$. By inspection, there are ten ways of getting from $A_0(n)$ to $A_0(n-1)$ and five ways each of getting from $A_0(n)$ to $A_1(n-1)$, $A_1(n)$ to $A_0(n-1)$, $A_1(n)$ to $A_1(n-1)$. Thus by repeatedly chopping the first two letter of w and the appropriate prefix of u, and recording the transitions, and the new 'state', each member of $A_0(n)$ or $A_1(n)$ induces an n-step walk on the above directed graph that starts at A_0 or A_1, respectively.

In the same vein, but even simpler, $B_0(n)$ and $B_1(n)$ correspond to n-step walks staring at A_0 and A_1 on that very same directed graph. The details are left to the reader.

REFERENCE

1. A. Benjamin and J. Quinn, "*Proofs that Really Count: The Art of Combinatorial Proofs*", The Math. Assoc. of Amer. 2003.

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