Appendix: Solution to Question 1

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Note Added Oct. 31, 2018: A few hours after the posting of this article in the arxiv, Will Sawin solved Question 1. A donation to the OEIS, in his honor, has been made. Here is his email message, reproduced with his kind permission.

I present a solution to question 1 of your recent arXiv preprint. Indeed, the constant is $e^{-\gamma}$. You ask about the series

$$\sum_{n=0}^{\infty} b(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-\frac{q^n}{n}} \quad .$$

Let us choose $c(n)$ so that we have

$$\sum_{n=0}^{\infty} c(n)q^n = (1-q)^2 \prod_{n=1}^{\infty} \frac{1}{1-\frac{q^n}{n}} \quad .$$

Using the formal identity

$$\frac{1}{1-q} = \prod_{n=1}^{\infty} e^{\frac{q^n}{n}} \quad ,$$

we have

$$(1-q)^2 \prod_{n=1}^{\infty} \frac{1}{1-\frac{q^n}{n}} = (1-q) \prod_{n=2}^{\infty} \frac{1}{1-\frac{q^n}{n}} = e^{-q} \prod_{n=2}^{\infty} \frac{e^{-\frac{q^n}{n}}}{1-\frac{q^n}{n}} \quad .$$

Now using the corollary

$$\frac{1}{1-\frac{q^n}{n}} = \prod_{d=1}^{\infty} e^{(\frac{q^n}{n})^d/d} \quad ,$$

of the same formal identity, we see that the terms in the infinite product have nonnegative coefficients.

Hence to show that $\sum_{n=0}^{\infty} |c(n)|$ is bounded, it suffices to show that the product of the sums of the coefficients of the terms in the infinite product are bounded, which is the same as showing that the infinite product is bounded with q substituted for 1.

If so, then $\sum_{n=0}^{\infty} c(n)$ is well-defined, and is equal to the product with q substituted for 1.

This is

$$e^{-1} \prod_{n=2}^{\infty} \frac{e^{-1/n}}{1-1/n} = \lim_{m\to\infty} \left[\exp(\sum_{n=1}^{m} -\frac{1}{n}) \prod_{n=2}^{m} \frac{1}{1-\frac{1}{n}} \right] = \lim_{m\to\infty} \left[\left(\exp(\sum_{n=1}^{m} -\frac{1}{n}) \right) \cdot m \right]$$

$$= \lim_{m\to\infty} \exp(\log m - \sum_{n=1}^{m} \frac{1}{n}) = e^{-\gamma} \quad .$$

So we know that $\sum_{n=1}^{\infty} c(n)$ converges absolutely to $e^{-\gamma}$. Hence the coefficients of

$$\frac{1}{1-q} \sum_{n=1}^{\infty} c(n) q^n \quad ,$$

which are the partial sums of this series converge, as n goes to infinity, to $e^{-\gamma}$.

Hence the coefficients of $\sum_{n=1}^{\infty} b(n)$, which are the partial sums of that series, are asymptotic to $n\, e^{-\gamma}$. $\square$