A 21st-Century Proof Of Dougall's Hypergeometric Sum Identity

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My guess is that, within fifty or hundred years (or it might be one hundred and fifty) computers will successfully compete with the human brain in doing mathematics, and that their mathematical style will be rather different from ours. Fairly long computational verifications (numerical or combinatorical) will not bother them at all, and this should lead not just to different sorts of proofs, but more importantly to different sorts of theorems being proved.

—David Ruelle[3].

Dougall's identity[2] is one of the most beautiful and general results in the theory of hypergeometric series, and Dougall's original proof([1], 5.1) is short and elegant. In this note we present a new proof that is even shorter, modulo purely routine verifications (that are easily carried out with computer algebra.) It was generated by the first author using an algorithm written by the second author[4].

Theorem (Dougall[2]; [1], p. 26 and section 5.1):

For any complex number z, and any integer k, let $(z)_k = z(z+1)...(z+k-1)$. The following identity is true for any positive integer n.

$$\sum_k (-1)^k \binom{n}{k} \cdot$$

$$\frac{(1+2a-b-c-d)_k(3/2+a-b/2-c/2-d/2)_k(1+a-c-d)_k(1+a-b-d)_k(1+a-b-c)_k(a+n)_k}{(1/2+a-b/2-c/2-d/2)_k(1+a-b)_k(1+a-c)_k(1+a-d)_k(2+a-b-c-d-n)_k(2+2a-b-c-d+n)_k}$$

$$= \frac{(2+2a-b-c-d)_n(b)_n(c)_n(d)_n}{(-1-a-b-c-d)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n} .$$

Proof: Let $R(n)$ and $F(n,k)$ be the sum and the summand respectively on the left. Since the theorem is obviously true for $n=0$, it would follow by induction once we show that

$$\begin{aligned}(-1-a-b-c-d+n)(1+a-b+n)(1+a-c+n)(1+a-d+n)R(n+1)\\ -(2+2a-b-c-d+n)(b+n)(c+n)(d+n)R(n)=0\end{aligned} \tag{1}$$

Let $G(n,k) := -(k+1+a-b-c)(k+1+a-b-d)\cdot$

$$\frac{(k+1+2a-b-c-d)(a+n+k)(k+1+a-c-d)(2n+a+1)(2+2a-b-c-d+n)}{((2k+2a-b-c-d+1)(2+2a+n+k-b-c-d)(a+n))}F(n,k)$$

It is readily verified (by machine, if possible) that

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\begin{aligned}
&(-1-a-b-c-d+n)(1+a-b+n)(1+a-c+n)(1+a-d+n)F(n+1,k)\\
&-(2+2a-b-c-d+n)(b+n)(c+n)(d+n)F(n,k) = G(n,k)-G(n,k-1)\ . \qquad (2)
\end{aligned}
$$

[Dividing (2) by $F(n,k)$ yields a routinely verifiable identity among rational functions].

and we get (1) by summing (2) with respect to k. QED

References

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2. J. Dougall, *On Vandermonde's theorem and some more general expansions*, Proc. Edinburgh Math. Soc., **25**(1907), 114-132.

3. Ruelle (David), *Is our mathematics natural? the case of equilibrium statistical mechanics*, Bull. (New Series) Amer. Math. Soc. **19** (1988), 259-268.

4. D.Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, submitted.