Proof of a Conjecture of Philippe Di Francesco and Paul Zinn-Justin related to the qKZ equation and to Dave Robbins' Two Favorite Combinatorial Objects

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For any Laurent polynomial $P(z_1, \dots, z_n)$ in the variables $(z_1, \dots, z_n)$, the *anti-symmetrizer*, $\mathcal{A}(P)$, is defined by:

$$\mathcal{A}(P)(z_1, \dots, z_n) = \sum_{\pi \in S_n} (-1)^{inv(\pi)} P(z_{\pi(1)}, \dots, z_{\pi(n)}) \quad ,$$

where S_n is the group of permutations on $\{1, \dots, n\}$ and $inv(\pi)$ is the number of inversions of π (the number of pairs $1 \leq i < j \leq n$ such that $\pi(i) > \pi(j)$).

Define the *negativizer* of P, $\mathcal{N}(P)$, to be the polynomial in $z_1^{-1}, \dots, z_n^{-1}$ obtained by removing all monomials that have at least one positive exponent.

For example, $\mathcal{A}(5 + z_1^{-1} + z_1^3 z_2^{-2}) = z_1^{-1} + z_1^3 z_2^{-2} - z_2^{-1} - z_2^3 z_1^{-2}$, and $N(5 + z_1^{-1} + z_1^3 z_2^{-2}) = 5 + z_1^{-1}$.

Let

$$F_n(z_1, \dots, z_n) = \prod_{i=1}^{n} z_i^{-2i+2} \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j) \prod_{1 \leq i < j \leq n} (1 + z_i z_j + T z_j) \quad ,$$

$$B_n(z_1, \dots, z_n) = \prod_{1 \leq i < j \leq n} (z_j^{-1} - z_i^{-1})(T + z_i^{-1} + z_j^{-1}) \quad .$$

In [1] (Eq. (4.4)), the following conjecture was made:

$$\mathcal{N}\left[\mathcal{A}\left[F_n(z_1, \dots, z_n)\right]\right] = B_n(z_1, \dots, z_n) \quad . \qquad (DiFZJ)$$

Let $G_n = \mathcal{A}(F_n)$. By breaking-up the summation over S_n into those π for which $\pi(n) = j$ $(j = 1, \dots, n)$, G_n is seen to satisfy the recurrence

$$G_n(z_1, \dots, z_n) = \sum_{j=1}^{n} (-1)^{n-j} \left(z_j^{1-2n} (1 - z_j^2) \prod_{i=1, i \neq j}^{n} \left[(1 + z_i z_j + T z_j)(1 - z_i z_j)\right] \right) \cdot$$

$$G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \quad .$$

Let $H_n = \mathcal{N}(G_n)$. We want to prove that for every n, $H_n = B_n$. We will do it by induction on n. The case $n = 1$ is obviously true (check!). Applying the $\mathcal{N}$-operation to both sides, and noting that since the factor in front of $G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ only has *positive* exponents in $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$,

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we have

$$H_n(z_1,\dots,z_n)=\mathcal{N}\Big\{\sum_{j=1}^{n}(-1)^{n-j}z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+z_iz_j+Tz_j)(1-z_iz_j)]\cdot$$

$$H_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)\Big\} \quad .$$

By the inductive hypothesis $H_{n-1}=B_{n-1}$ so we have to prove that

$$B_n(z_1,\dots,z_n)=\mathcal{N}\Big\{\sum_{j=1}^{n}\left((-1)^{n-j}z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+z_iz_j+Tz_j)(1-z_iz_j)]\right)\cdot$$

$$B_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)\Big\} \quad .$$

It turns out that the *negativizand* on the right side is *nice*:

$$\sum_{j=1}^{n}\left((-1)^{n-j}z_j^{1-2n}(1-z_j^2)\prod_{i=1,i\neq j}^{n}[(1+Tz_j+z_iz_j)(1-z_iz_j)]\right)\cdot B_{n-1}(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)=$$

$$B_n(z_1,\dots,z_n)(1-z_1^2z_2^2\cdots z_n^2) \quad . \qquad (NiceSurprise)$$

We now need:

A Simple Fact: Each and every monomial in $B_n(z_1,\dots,z_n)z_1^2z_2^2\cdots z_n^2$ has at least one positive exponent, hence its negativizer is 0.

Proof of the Simple Fact: (This simple argument of referee Philippe Di Francesco replaces a previous more complicated argument).

$$B_n(z_1,\dots,z_n)=\prod_{1\le i<j\le n}(Tz_j^{-1}+z_j^{-2}-Tz_i^{-1}-z_i^{-2})=\prod_{1\le i<j\le n}[(T/2+z_j^{-1})^2-(T/2+z_i^{-1})^2] \quad .$$

So each term in the last product can't have all the variables, and the same is true when it is fully expanded, and it follows that each monomial in $B_n(z_1,\dots,z_n)z_1^2z_2^2\cdots z_n^2$ has at least one positive exponent.

Hence:

$$\mathcal{N}(B_n(z_1,\dots,z_n)(1-z_1^2z_2^2\cdots z_n^2))=B_n(z_1,\dots,z_n) \quad ,$$

and $(DiFZJ)$ would follow. It remains to prove $(NiceSurprise)$. By dividing both sides by B_n, letting $z_i\to z_i^{-1}$ $(1\le i\le n)$, and rearranging terms, we see that it is equivalent to the following:

$$\sum_{j=1}^{n}\frac{(2z_j+T)\prod_{i=1}^{n}[(1+z_i(T+z_j))(1-z_iz_j)]}{(1+z_jT+z_j^2)\prod_{i=1,i\neq j}^{n}(z_j-z_i)\prod_{i=1}^{n}(z_j+z_i+T)}=(-1)^{n-1}(1-z_1^2\cdots z_n^2) \quad .$$

$$(IntriguingIdentity)$$

Let $\alpha = \alpha(T)$ and $\beta = \beta(T)$ be the two roots of $1 + zT + z^2 = 0$. Note that $\alpha + \beta = -T$ and $\alpha\beta = 1$.

Recall the *Lagrange Interpolation Formula*. For any polynomial $P(z)$ of degree $\leq N-1$ and *any* numbers $w_1, \ldots, w_N$, we have

$$P(z) = \sum_{j=1}^{N} \frac{P(w_j) \prod_{i=1, i \neq j}^{N} (z - w_i)}{\prod_{i=1, i \neq j}^{N} (w_j - w_i)} \quad . \tag{LI}$$

[Proof: both sides are equal at the N numbers $z = w_1, \ldots, w_N$, hence they are *always* equal (being polynomials of degree $\leq N-1$)].

In fact, we will only need the corollary obtained by comparing the coefficient of z^{N-1} on both sides

$$Coefficient \quad of \quad z^{N-1} \quad in \quad P(z) = \sum_{j=1}^{N} \frac{P(w_j)}{\prod_{i=1, i \neq j}^{n} (w_j - w_i)} \quad .$$

Now let

$$P(z) := (2z + T) \prod_{i=1}^{n} [(1 + z_i(T + z))(1 - z_i z)] \quad ,$$

and pick the w's to be the $2n+2$ numbers

$$\{z_1, \ldots, z_n, \quad -z_1 - T, \ldots, -z_n - T, \quad \alpha, \beta\} \quad ,$$

in order to get:

$$2(-1)^n z_1^2 \cdots z_n^2 = \sum_{j=1}^{n} \frac{P(z_j)}{(1 + z_j T + z_j^2) \prod_{i=1, i \neq j}^{n} (z_j - z_i) \prod_{i=1}^{n} (z_j + z_i + T)}$$

$$+ \sum_{j=1}^{n} \frac{P(-T - z_j)}{(1 + z_j T + z_j^2) \prod_{i=1}^{n} (-z_j - T - z_i) \prod_{i=1, i \neq j}^{n} (-z_j - T + z_i + T)}$$

$$+ \frac{P(\alpha)}{(\alpha - \beta) \prod_{i=1}^{n} (\alpha - z_i) \prod_{i=1}^{n} (\alpha + z_i + T)}$$

$$+ \frac{P(\beta)}{(\beta - \alpha) \prod_{i=1}^{n} (\beta - z_i) \prod_{i=1}^{n} (\beta + z_i + T)} \quad .$$

Using $P(-T - z_j) = -P(z_j)$ (check!), $\alpha\beta = 1$ and $\alpha + \beta = -T$, and manipulating the obvious cancellations, and observing that the first two terms are identical, while the sum of the last two terms simplifies to $2(-1)^n$, we get that ($IntriguingIdentity$) is indeed true. $\Box$.

Reference

1. P. Di Francesco and P. Zinn-Justin, *Quantum Knizhnik-Zamolodchikov equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices*, `http://www.arxiv.org/abs/math-ph/0703015` .