All the Moments of the Vertex Degrees of Randomly Generated Graphs

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Consider the set of all labelled graphs on n vertices and m edges. By the handshaking lemma the average degree of *every* graph is $2m/n$. Higher moments do depend on the graph, but one can easily compute the average of the average of $\binom{deg(v)}{k}$, for any k. Let $N = n(n-1)/2$. The answer is $\binom{n-1}{k}\binom{N-k}{m-k}/\binom{N}{m}$, which equals the rational funcion (in n and m) $\binom{n-1}{k}(m-k+1)_k/(N-k+1)_k$, where as usual $(a)_k := a(a+1)\cdots(a+k-1)$. Indeed, for any particular vertex v, the sum of $\binom{deg(v)}{k}$ over all graphs with n vertices and m edges counts such graphs with a k-subset of v's neighbors marked. Clearly the number of these marked graphs is $\binom{n-1}{k}\binom{N-k}{m-k}$, (the first term for choosing the marked neighbors and the second term for choosing the remaining edges. From this it is easy to compute any desidred moment.

The special case $k = 2$ yields a one-line proof of the result in **Ivan Gutman and Peter Paule, Univ. Beograd. Publ. Elek. Fak. 13 (2002), 30-35**. [`http://mathematika.etf.bg.ac.yu/publikacije/`].

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