Cutting 4 by n grids into two congruent pieces

Robert Dougherty-Bliss, Natalya Ter-Saakov, and Doron Zeilberger

Pour Jean-Paul Delahaye, avec admiration

Introduction

In the March 2025 issue of *Pour La Science* (the French analog of Scientific American), in his delightful monthly column [D], Jean-Paul Delahaye described in detail the solution of the following counting problem.

In how many ways can you cut a $3 \times 2n$ rectangle consisting of $6n$ unit squares (in other words a $3 \times 2n$ (colorless) checkerboard) into two connected **congruent** *pieces?*

In collaboration with his *épouse*, Martine Raison, they used human ingenuity to prove that this number is given **explicitly** by the beautiful formula $2^{n+1} - n - 1$.

We asked ourselves, what about a $4 \times n$ checkerboard? Now human ingenuity is still *necessary*, but (most probably) not *sufficient*.

In collaboration with our beloved silicon friends we proved the following deep theorem.

Theorem 1: Let c_n be the number of ways of cutting a $4 \times n$ grid into two (connected) congruent pieces, then the (ordinary) generating function is:

$$\sum_{n=0}^{\infty} c_n x^n = \frac{x\left(2x^8 - 4x^7 + 8x^6 - 7x^5 + 3x^4 + 2x^3 - 5x^2 + x + 1\right)}{(x-1)^2\left(x^4 + 3x^2 - 1\right)\left(x^4 + 2x^2 - 1\right)} \quad .$$

For the sake of the OEIS the first 30 terms are:

$$1, 3, 5, 14, 22, 54, 84, 197, 305, 696, 1075, 2410, 3716, 8231, 12676, 27844, 42843, 93558, 143865, 312859, 480868,$$
$$1042624, 1602002, 3466064, 5324385, 11501987, 17665729, 38119718, 58540246, 126217718 \quad .$$

To our delight, this sequence was not (viewed Sept. 4, 2025) in the OEIS, and we hope to submit it soon.

A convenient way to represent such *decoupages* into congruent pieces is via $4 \times n$ $0-1$ matrices, where because of symmetry we assume that the last entry of the first row is always 1. There are $\binom{4n-1}{2n-1}$ such candidate matrices (without the conditions of connectivity and congruence), but most of them do not obey the condition. From now on we will call them *good martices*.

Here is one of the 54 such 4×6 good matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad ,$$

and here is one of the 696 such 4×10 good matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad .$$

To see all good matrices $4 \times n$ matrices for $1 \leq n \leq 12$ see the output file:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oDecoupage1.txt` .

It is easy to see that any such good matrix must start with a zeroes followed by $n - a$ ones, where $0 \leq a \leq \lfloor n/2 \rfloor$. We also have the more detailed theorem.

Theorem 2: Let $f_a(x)$ be the coefficient of z^a in the Maclaurin expansion (with respect to the variable z) of the rational function

$$\frac{-x^3 - x^2 + x + 1 + \left(x^5 + 2x^3 - x^2 - x\right) z + \left(x^4 - x^3 + 2x^2 - 1\right) z^2}{z^3 + z^2 - 3z + 1}$$

then the number of $4 \times n$ good matrices whose first row starts with a consecutive zeroes (and hence ends with $n - a$ consecutive ones) is the coefficient of x^n in the Maclaurin expansion of the rational function

$$\frac{x^{2a} f_a(x)}{(x-1)\left(x^4 + 3x^2 - 1\right)} \quad .$$

Note that if we drop the condition of the two pieces being congruent, and consider only square matrices, then we have the (computationally) challenging problem of the *Gerrymeander* sequence [Sl] , beautifully treated in [KKS] and extended in [GJ].

Maple Package

This article is accompanied by a Maple package `Decoupage.txt` that can be gotten from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/Decoupage.txt` .

The front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/decoupage.html` contains some sample output files.

How did we find this beautiful Generating function?

We asked our computer to show us all these creatures up to $n = 12$. We stared at them and tried to guess a *grammar*. It was very reasonable to expect a *type* 3 grammar, also known as *finite state machine*, with *starting* states and *accepting* states, and 'transitions edges' in a certain finite alphabet. Once we had the grammar, we (or rather our computer) could easily find the *transfer* *matrix* and, in turn, using *symbolic computation*, usig that very useful took-of-the-trade called the *transfer matrix method* ([St], section 4.7) derive the desired generating function.

We will now actually describe the grammar.

NATASHA and ROBERT YOUR PART

Future work

What about the analogous problem of $5 \times 2n$, $6 \times n$, $7 \times 2n$, $8 \times n$ checkerboards? Eventually it would be nice to have the computer do everything, i.e. discover the grammar, and then process it *all by itself* without any need of human ingenuity. But even computers can only go so far, and we are sure that humankind, and even computerkind will **never** have the generating function (most probably a rational function) for the sequence counting $30 \times n$ good matrices.

References

[D] Jean-Paul Delahaye, *Combinatoire pour les rectangles*, *Logique & Calcul*, **Pour La Science** No. **569**, Mars 2025.

[GJ] Anthony J. Guttmann and Iwan Jensen, *The gerrymander sequence, or A348456*, arXiv:2211.14482 [math.CO], 2022. `https://arxiv.org/abs/2211.14482` .

[KKS] Manuel Kauers, Christoph Koutschan, and George Spahn, *How Does the Gerrymander Sequence Continue?*, J. Int. Seq., **25** (2022), Article 22.9.7.
`https://cs.uwaterloo.ca/journals/JIS/VOL25/Kauers/kauers6.html`.
arxiv version: `https://arxiv.org/abs/2209.01787` .

[Sl] Neil Sloane, OEIS sequence A348456 `https://oeis.org/A348456`.

[St] Richard P. Stanley, *"Enumerative Combinatorics"*, Volume 1, Cambridge University Press, First Edition, Wadsworth and Brooks/Cole, 1986.

Robert Dougherty-Bliss, Department of Mathematics, Dartmouth College, 29 N. Main Street, 6188 Kemeny Hall, Hanover NH 03755-3551 . Email: `robert dot w dot bliss at gmail dot com`

Natalya Ter-Saakov, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `nt399 at rutgers dot edu` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .

Sept. 10, 2025