A combinatorial proof of Cramer's Rule

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Dedicated to my Rutgers colleague Antoni A. Kosinski (b. May 25, 1930) who proved that Cramer's Rule is indeed due to Cramer.

Prerequisites: We assume that readers are familiar with the notions of : *integer* (e.g. 3, 11), *set* (e.g. $\{1,2,3\}$), *addition* (of numbers or symbols) (e.g. $a+b$) , *multiplication* (e.g. bc), and "$>$, $=$, $<$, $\leq$, $\geq$". Two symbols a and b *commute* if $ab = ba$. All the symbols in this paper commute. No other knowledge is needed!

Notation: If w is a *weight* defined on a set S (i.e. a way of assigning a number, or algebraic expression to members of S), then $w(S) := \sum_{s\in S} w(s)$. For example if $S = \{1,4\}$, $w(1) = a$, $w(4) = b$, then $w(\{1,4\}) = a + b$.

Definitions

• For any positive integer n, an n-permutation is a list of integers $\pi = \pi_1 \dots \pi_n$, where $1 \leq \pi_i \leq n$ for all $1 \leq i \leq n$ and $\pi_i \neq \pi_j$ if $i \neq j$.

• The set of all n-permutations is denoted by S_n. For example, $S_3 = \{123, 132, 213, 231, 312, 321\}$.

• For an n-permutation $\pi = \pi_1 \dots \pi_n$, a pair (i, j), where $1 \leq i < j \leq n$, is an *inversion* if $\pi_i > \pi_j$. For example, If $\pi = 51423$ then $(1,2), (1,3), (1,4), (1,5), (3,4), (3,5)$ are inversions of π. Let $inv(\pi)$ be the *number of inversions*, for example $inv(51423) = 6$.

• Let n be a positive integer, and let $a_{i,j} (1 \leq i, j \leq n)$ and b_i $(1 \leq i \leq n)$ be $n^2 + n$ *commuting* symbols (or numbers). Define $n + 1$ *weights* w_j $(0 \leq j \leq n)$ on S_n as follows:

$$w_0(\pi) := (-1)^{inv(\pi)} \prod_{1\leq k\leq n} a_{\pi_k,k} \quad ,$$

$$w_j(\pi) := (-1)^{inv(\pi)}\, b_{\pi_j} \prod_{\substack{1\leq k\leq n\\ k\neq j}} a_{\pi_k,k} \quad (1 \leq j \leq n) \quad .$$

Theorem (Cramer [C]): Let

$$X_j := w_j(S_n) \quad (0 \leq j \leq n) \quad ,$$

then

$$x_j := \frac{X_j}{X_0} \quad (1 \leq j \leq n) \quad ,$$

satisfy the n linear equations

$$\sum_{j=1}^{n} a_{i,j} x_j \;=\; b_i \quad (1 \leq i \leq n) \quad . \qquad (C_i)$$

Combinatorial Proof: By multiplying (C_i) by X_0, we have to prove

$$\sum_{j=1}^{n} a_{i,j} X_j \;=\; b_i X_0 \quad (1 \le i \le n) \quad . \tag{C_i'}$$

For any positive integer n, let F_n be the following set (with $n \cdot n!$ members, where $n! := 1 \cdot 2 \cdots n$)

$$F_n := \{[j,\pi]\,;\, 1 \le j \le n \quad , \quad \pi \in S_n\} \quad .$$

For any i $(1 \le i \le n)$, define a weight W_i on F_n as follows:

$$W_i([j,\pi]) := a_{i,j} w_j(\pi) = a_{i,j} (-1)^{inv(\pi)} b_{\pi_j} \prod_{\substack{1 \le k \le n \\ k \ne j}} a_{\pi_k,k} \quad .$$

The left side of (C_i') is $W_i(F_n)$.

Definition: $[j,\pi] \in F_n$ is an i-*good guy* if $\pi_j = i$, otherwise it is an i-*bad guy*. Let $G_{n,i}$ and $B_{n,i}$ be the subsets of F_n consisting of the i-good guys and i-bad guys respectively.

Obviously, since $F_n = G_{n,i} \cup B_{n,i}$, we have

$$W_i(F_n) \;=\; W_i(G_{n,i}) + W_i(B_{n,i}) \quad .$$

Fact 1:

$$W_i(G_{n,i}) = b_i\, X_0$$

Proof of Fact 1: If $[j,\pi] \in F_n$ is a good guy then, since $\pi_j = i$, we have:

$$W_i([j,\pi]) = (-1)^{inv(\pi)}\, a_{i,j}\, b_{\pi_j} \prod_{\substack{1 \le k \le n \\ k \ne j}} a_{\pi_k,k}$$

$$= (-1)^{inv(\pi)}\, a_{\pi_j,j}\, b_i \prod_{\substack{1 \le k \le n \\ k \ne j}} a_{\pi_k,k} = (-1)^{inv(\pi)}\, b_i \prod_{1 \le k \le n} a_{\pi_k,k} \;=\; b_i w_0(\pi) \quad .$$

Hence $W_i(G_{n,i}) = b_i w_0(S_n) = b_i X_0$. $\square$

Fact 2:

$$W_i(B_{n,i}) = 0 \quad .$$

Proof of Fact 2: Let $[j,\pi] \in F_n$ be an i-bad guy. Let $a := \pi_j$ and $j' := \pi^{-1}(i)$. Of course $a \ne i$ and $j' \ne j$. Define a permutation σ by transposing $\pi_j = a$ and $\pi_{j'} = i$, in other words $\sigma_j = i$, $\sigma_{j'} = a$ and $\sigma_k = \pi_k$ if $k \notin \{j, j'\}$. Let

$$T_i([j,\pi]) := [j', \sigma] \quad .$$

We have

$$W_i([j,\pi]) \; = \; (-1)^{inv(\pi)} \, a_{i,j} \, b_a \prod_{\substack{1 \leq k \leq n \\ k \neq j}} a_{\pi_k,k} = (-1)^{inv(\pi)} a_{i,j} \, b_a \, a_{\pi_{j'},j'} \prod_{\substack{1 \leq k \leq n \\ k \neq j,j'}} a_{\pi_k,k}$$

$$= (-1)^{inv(\pi)} \, a_{i,j} \, b_a \, a_{i,j'} \prod_{\substack{1 \leq k \leq n \\ k \neq j,j'}} a_{\pi_k,k} \quad .$$

Similarly

$$W_i([j',\sigma]) = (-1)^{inv(\sigma)} \, a_{i,j'} \, b_a \prod_{\substack{1 \leq k \leq n \\ k \neq j'}} a_{\sigma_k,k} = (-1)^{inv(\sigma)} \, a_{i,j'} \, b_a \, a_{\sigma_j,j} \prod_{\substack{1 \leq k \leq n \\ k \neq j,j'}} a_{\sigma_k,k}$$

$$= (-1)^{inv(\sigma)} \, a_{i,j'} \, b_a \, a_{i,j} \prod_{\substack{1 \leq k \leq n \\ k \neq j,j'}} a_{\sigma_k,k} \quad .$$

Since $inv(\pi) - inv(\sigma)$ is odd (why?), and π_k and σ_k coincide if $k \not\in \{j, j'\}$, we have, by *commutativity*, that for any i-bad guy b, $W_i(b) + W_i(T_i(b)) = 0$.

Since $T_i(T_i(b)) = b$ for all i-bad guys b (why?), all the bad guys can be arranged into mutually W_i-canceling pairs, proving Fact 2. $\Box$

Combining Facts 1 and 2, (C'_i), and hence Cramer's Rule, follow. $\Box$

Comments: **1.** There are several 'short' proofs of Cramer's rule that can be found in Wikipedia and its references, but they all assume knowledge of linear algebra. Our proof is fully *self-contained*, and does not assume *anything* besides the prerequisites listed at the beginning. We believe that if you include all the necessary background, our proof is the shortest.

2. For a fascinating defense of Gabriel Cramer's priority for his rule, see Antoni Kosinski's article [K].

References

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