Five More Proofs of the Cosine Addition Formula (Inspired by Mark Levi's Perpetuum Mobile Proof)

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Dedicated to my hero Mark Levi

Every time I get *SIAM News*, I immediately turn to Mark Levi's wonderful column *Mathematical Curiosities* `https://www.marklevimath.com/sinews`. In the current issue, Levi [L] gave a *proof from the book* of the trigonometric identity $\cos(\alpha+\beta) = \cos\beta\cdot\cos\alpha - \sin\beta\cdot\sin\alpha$, by showing that it follows from the non-existence of **Perpetual Motion**. This reminded me of five other proofs, none of them, admittedly, as nice as Mark Levi's proof (although the last one is a close second).

Pre-calculus: See the Wikipedia article *List of Trigonometric Identities*. $\square$

Linear Algebra: Rotating the vector $[1,0]^T$ by an angle of α gives the vector $[\cos\alpha, \sin\alpha]^T$. Rotating the vector $[0,1]^T$ by an angle of α gives the vector $[-\sin\alpha, \cos\alpha]^T$. By **linearity**, the **rotation matrix**, R_α is

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad .$$

Now use **matrix multiplication** and the fact that $R_{\alpha+\beta} = R_\alpha R_\beta$. $\square$

Complex Variable: Take the real parts of both sides of $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ $\square$

Differential Equations: Both sides satisfy the differential equation (viewed as a function of α) $y'' + y = 0$, subject to the initial conditions $y(0) = \cos\beta$, $y'(0) = -\sin\beta$. Now use uniqueness. $\square$

Combinatorics: $\cos\alpha$ (resp. $\sin\alpha$) is the **exponential generating function** of **increasing sequences of integers of even length** (resp. of **odd length**) with weight $(-1)^{length/2}$ (resp. $(-1)^{(length-1)/2}$), see [Z]. Hence $\cos(\alpha+\beta)$ is the exponential generating function of two-colored increasing sequences of even length, with say, colors α and β. If the number of integers colored α is even (resp. odd) we get the first term (resp. second term) on the right. $\square$

References

[L] Mark Levi, *Cosine Addition Formula and Perpetual Motion*, SIAM News **55 #3** (April 2022), p.7.

[Z] Doron Zeilberger, *Enumerative and Algebraic Combinatorics*, in: *"Princeton Companion to Mathematics"* , (Timothy Gowers, ed.), Princeton University Press,2008, 550-561.
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/enuPCM.pdf` .

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