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# Counting Condorcet

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Abstract: We give an elegant bijective proof that the number of vote-count profiles that lead to the famous Condorcet paradox with three candidates and 2n-1 voters equals $(n+3)(n+2)(n+1)n(n-1)/60$. We then use this bijection to efficiently enumerate the total number of Condorcet voting profiles with a given number of (odd) voters, and related probabilities.

**Keywords**: Condorcet Paradox, linear recurrences, compositions, bijective proofs.

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# 1. Preface: Why yet another paper on the Condorcet Paradox?

One of the most fascinating paradoxes is the **Condorcet Paradox** [C] (see section 2.1 of [GM] for a very lucid and engaging account, and [BF], section 8, and [GF], for a comprehensive one). In its simplest form there are $n$ voters and three candidates (let's call them, unoriginally, 1,2, and 3 rather than the original *Pierre, Paul, and Jaques*). Each voter must decide on one of the six complete rankings

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321 \quad .$$

Hence there are altogether $6^n$ *voting profiles*. Suppose that $x_{123}$ voters picked the ranking 123, $x_{132}$ voters picked the ranking 132, $x_{213}$ voters picked the ranking 213, $x_{231}$ voters picked the ranking 231, $x_{312}$ voters picked the ranking 312, and $x_{321}$ voters picked the ranking 321.

So each voting profile leads, in a unique way to what we will call a *vote-count profile*.

$$[x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}] \quad .$$

For any two candidates $i$ and $j$, where $i \neq j$ and $1 \leq i, j \leq 3$ we say that $i$ beats $j$ if the number of voters who prefer $i$ to $j$ **strictly** exceeds the number of voters who prefer $j$ to $i$.

A **Condorcet scenario** happens when there is a *cycle* : 1 beats 2, 2 beats 3, and, surprise!, 3 beats 1, or vice versa.

A natural question is:

*"Out of the $6^n$ possible voting profiles, how many exhibit the Condorcet paradox?"*

In this article, we will describe how to compute these numbers in linear time and constant memory (linear-memory if you go by bit-size).

For example we will show how to compute **exactly** this number when there are 199 voters. It happens to be the following integer
6211370515760300852136923742146263204611276496186703609937916433495661887654521\
5323029575682542499317816686500703191544876674381121172728733754448942000 .

Hence the **probability** of such a scenario, assuming that each voter picks a ranking uniformly at random, and the choices are *independent*, is:

$$0.087314950043755200423273246852141061225904929801254 16\ldots$$

Similarly, almost as fast, we can get the **exact** number of Condorcet voting-profiles with 19999 voters. This happens to be a certain 15562-digit integer that is too large to reproduce here, but can be seen here:
`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oCondorcet3e.txt` .
It implies that the probability is

$$0.0877356075296129282604035628906239948940764715\ldots \quad .$$

Using asymptotics we can very accurately estimate the *limiting* probability, as $n$ goes to infinity. It agrees, to any desired accuracy, with Guilbaud's 1952 exact value (see [BF], p. 205)

$$\frac{1}{4} - \frac{3}{2\pi}\sin^{-1}(\frac{1}{3}) \;=\; 0.08773982805\ldots \quad .$$

In particular, the present paper will result in a new sequence in the OEIS, that starts as follows
$0, 12, 540, 21000, 785820, 28956312, 1058809752, 38545567632, 1399354322652, 50707958458872, 1835099465988360, 66348521294296176, 2397139928161319640,$
$86559958069097395440, 3124302168622853150640, 112729791393354644416800\ , \ldots$
Its description is 'number of Condorcet voting profiles with three candidates and $2n-1$ voters (starting with $n=1$)'.
To our surprise, this important sequence (viewed April 21, 2022) was not in the OEIS [S]. Neither did a search for 28956312 (the number with 11 voters) in **JSTOR** (that has 1879 articles containing the words 'Condorcet' and 'Paradox') return anything.
The Condorcet paradox is a precursor of Ken Arrow's famous *impossibility theorem* [A] that states that there does not exist *any* social choice function that is **always** guaranteed to give a clear-cut ranking of all the candidates, under some very reasonable set of "axioms". This theorem, that was the subject of a whole book [A], and that ultimately lead to a Nobel prize, with the benefit of hindsight, now has a one-page proof [Y].

# 2. Counting All Vote-Count Profiles

As mentioned above, a vote-count profile with $n$ voters (and three candidates) is a list of non-negative integers

$$[x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}] \quad ,$$

that add up to $n$, in other words, it is (what we call) a composition of $n$ into 6 parts.
**Definition**: A composition of $n$ into $k$ parts is a list $(x_1, \ldots, x_k)$ of $k$ **non-negative** integers that add-up to $n$.
(Note that the usual definition of *composition* requires that the entries are strictly positive. Of course there is a trivial bijection between our kind of compositions of $n$ into $k$ parts and the usual kind of compositions of $n+k$ into $k$ parts.)
We first need a simple very well-known lemma.
**Lemma 1**: The number of compositions (in our sense) of $n$ into $k$ parts is

$$\binom{n+k-1}{k-1} \quad .$$

Let's recall three proofs.
**First Proof**: The (ordinary) generating function of the set of *non-negative* integers is $1+x+x^2+\cdots = \frac{1}{1-x}$. Hence the generating function of compositions into $k$ parts is $(\frac{1}{1-x})^k = (1-x)^{-k}$. Hence the number of composition of $n$ is the coefficient of $x^n$ in $(1-x)^{-k}$, in other words

$$(-1)^n\binom{-k}{n} \;=\; (-1)^n\frac{(-k)(-k-1)\cdots(-k-(n-1))}{n!} \;=\; \frac{k(k+1)\cdots(k+n-1)}{n!} \;=\; \binom{n+k-1}{k-1} \quad .$$

**Second Proof**: Let $A(n,k)$ be the number of compositions of $n$ into $k$ non-negative integers. Let $(x_1, \ldots, x_k)$ be such a composition. If $x_k = 0$ then it is equi-numerous with $A(n, k-1)$. If $x_k \geq 1$ then mapping it to $(x_1, \ldots, x_{k-1}, x_k - 1)$ gives an element counted by $A(n-1, k)$. Hence

$$A(n,k) \;=\; A(n,k-1) \;+\; A(n-1,k) \quad .$$

Now use induction to prove that $A(n,k) = \binom{n+k-1}{k-1}$ .
**Third Proof**: The mapping

$$(x_1, \ldots, x_k) \rightarrow \{x_1+1, x_1+x_2+2, \ldots, x_1+x_2+\cdots+x_{k-1}+k-1\} \quad ,$$

is a bijection to the family of $(k-1)$-element subsets of $\{1, 2, \ldots, n+k-1\}$ .

Hence we have the following corollary.

**Corollary**: The **total** number of **vote-count profiles** with three candidates and $2n-1$ voters is

$$\binom{2n+4}{5} \quad .$$

# 3. Counting Condorcet Vote-Count Profiles

Given a vote-count profile $[x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}]$, when does it produce a $1 \to 2 \to 3 \to 1$ cycle?

1 beats 2 if

$$x_{123} + x_{132} + x_{312} > x_{213} + x_{231} + x_{321} \quad ; \tag{C12}$$

2 beats 3 if

$$x_{123} + x_{213} + x_{231} > x_{132} + x_{312} + x_{321} \quad ; \tag{C23}$$

3 beats 1 if

$$x_{231} + x_{321} + x_{312} > x_{213} + x_{123} + x_{132} \quad . \tag{C31}$$

So we have to count, out of all the compositions of $2n-1$ into 6 parts those that satisfy the above three inequalities (and at the end double them to account for the reverse cycle).

We will now prove the following crucial theorem, and give an elegant bijective proof.

**Theorem 1**: The number of **Condorcet vote-count profiles** with three candidates and $2n-1$ voters is

$$2\binom{n+3}{5} \quad .$$

**Proof**: Consider the affine-linear mapping from compositions of $n-2$ into 6 parts to 1231 Condorcet vote-count profiles

$$[x_1, x_2, x_3, x_4, x_5, x_6] \to [x_1 + x_4 + x_6 + 1, x_2, x_3, x_2 + x_4 + x_5 + 1, x_3 + x_5 + x_6 + 1, x_1] \quad .$$

In other words

$$x_{123} \,=\, x_1 + x_4 + x_6 + 1 \quad , \quad x_{132} \,=\, x_2 \quad , \quad x_{213} \,=\, x_3 \quad ,$$
$$x_{231} \,=\, x_2 + x_4 + x_5 + 1 \quad , \quad x_{312} \,=\, x_3 + x_5 + x_6 + 1 \quad , \quad x_{321} \,=\, x_1 \quad .$$

It is readily seen that indeed

$$x_{123} + x_{132} + x_{213} + x_{231} + x_{312} + x_{321} \,=\, 2n-1 \quad ,$$

and that the three inequalities $(C12)$, $(C23)$ and $(C31)$ hold. Indeed

$$(x_{123} + x_{132} + x_{312}) - (x_{213} + x_{231} + x_{321}) \,=\, 2x_6 + 1 > 0 \quad ,$$
$$(x_{123} + x_{213} + x_{231}) - (x_{132} + x_{312} + x_{321}) \,=\, 2x_4 + 1 > 0 \quad ,$$
$$(x_{231} + x_{321} + x_{312}) - (x_{213} + x_{123} + x_{132}) \,=\, 2x_5 + 1 > 0 \quad ,$$

since all the $x_i$ are (by definition) non-negative.

The inverse bijection is

$$[x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}] \to [x_1, x_2, x_3, x_4, x_5, x_6] \quad .$$

where

$$x_1 = x_{321} \quad , \quad x_2 = x_{132} \quad , \quad x_3 = x_{213} \quad ,$$
$$x_4 \,=\, \frac{(x_{123} + x_{213} + x_{231}) - (x_{132} + x_{312} + x_{321}) - 1}{2} \quad ,$$
$$x_5 = \frac{(x_{231} + x_{321} + x_{312}) - (x_{213} + x_{123} + x_{132}) - 1}{2} \quad ,$$
$$x_6 = \frac{(x_{312} + x_{132} + x_{123}) - (x_{321} + x_{231} + x_{213}) - 1}{2} \quad .$$

Note that $x_1, x_2, x_3$ are non-negative, of course, and so are $x_4, x_5, x_6$ thanks to the three inequalities. Also adding them gives $n-2$ (check!) .

It is also readily seen that these mappings are inverses of each other. Hence the number of Condorcet vote-count profiles with $2n-1$ voters and the cycle 1231 equals the number of compositions of $n-2$ into 6 parts, that by the above lemma equals $\binom{n-2+5}{5} = \binom{n+3}{5}$. By symmetry the number of vote-count profiles with the reverse cycle also equals this, so doubling proves the theorem.

**Corollary**: If every vote-count profile is equally likely, the probability of it being Condorcet is

$$\frac{2\binom{n+3}{5}}{\binom{2n+4}{5}} \quad ,$$

that converges to $\frac{1}{16} = 0.0625$ as $n \to \infty$.

**Counting Condorcet Voting-Profiles**

Each **vote-count** profile with $2n-1$ voters

$$[x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}] \quad ,$$

gives rise to

$$\frac{(2n-1)!}{x_{123}!\, x_{132}!\, x_{213}!\, x_{231}!\, x_{312}!\, x_{321}!}$$

**voting-profiles** (where order matters).

What's nice about our bijective proof is that it gives us a 'parametric representation' of a typical Condorcet vote-count profile, and hence we immediately have the following proposition.

**Proposition 1**: Let $a(n)$ be the number of Condorcet voting-profiles with three candidates and $2n-1$ voters, then it is given by the following 5-fold sum:

$$a(n) = 2 \sum_{i_1=0}^{n-2} \sum_{i2=0}^{n-2-i_1} \sum_{i_3=0}^{n-2-i_1-i_2} \sum_{i4=0}^{n-2-i_1-i_2-i_3} \sum_{i_5=0}^{n-2-i_1-i_2-i_3-i_4}$$

$$\frac{(2n-1)!}{(n-1-i_2-i_3-i_5)!i_2!i_3!\,(i_2+i_4+i_5+1)!\,(n-1-i_1-i_2-i_4)!i_1!} \quad .$$

It follows from **Wilf-Zeilberger** theory [PWZ], and more specifically from the mutli-variable Zeilberger algorithm, that there **exists** a linear recurrence with polynomial coefficients. It can be easily found algorithmically [AZ], but it is even easier to find it by 'guessing' that can be made fully rigorous *a posteriori*. This leads to the next theorem.

**Theorem 2**: The number of Condorcet voting profiles with three candidates and $2n-1$ voters, let's call it $a(n)$, satisfies the following third-order recurrence

$$a(n) \;=\; \frac{4\left(19n^2-57n+45\right)}{(n-1)^2}\cdot a(n-1) - \frac{36\,(2n-3)\left(22n^2-99n+111\right)}{(n-2)\,(n-1)^2}\cdot a(n-2)$$

$$+\frac{1296\,(n-3)\,(2n-3)\,(2n-5)}{(n-2)\,(n-1)^2}\cdot a(n-3) \quad ,$$

subject to the initial conditions $a(1) = 0$, $a(2) = 12$, and $a(3) = 540$.

Using this recurrence we can compute many terms! See

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oCondorcet3b.txt` .

**Restricted Choices**

Suppose that instead of allowing all six rankings, every voter has to pick from the restricted set $\{123, 231, 312\}$. Using similar reasonings we have the following theorem.

**Theorem 3**: The number of Condorcet voting-profiles where all the choices are from $\{123, 231, 312\}$, let's call it $b(n)$, satisfies the second-order linear recurrence

$$\frac{36\,(2n+1)\,b(n)}{n+1} - \frac{(17n+13)\,b(n+1)}{n+1} + b(n+2) = 0 \quad ,$$

subject to the initial conditions $b(1) = 0$ and $b(2) = 6$.

For the sake of the OEIS, the first 12 terms:

$$0, 6, 90, 1050, 11130, 112266, 1099098, 10550826, 99899514, 936435786, 8711707290, 80572452714 \quad .$$

**Loaded Choices**

If each voter, *independently*, rolls a six-faced die marked with the six permutations of $\{1, 2, 3\}$ where the probability of the ranking $\pi$ is $p_\pi$ then, again thanks to the bijection we have the following proposition.

**Proposition 2**: If each of the $2n-1$ voters, picks, independently, one of the six ranking with distribution

$$[p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}] \quad ,$$

(where of course they add-up to 1), then the probability of a Condorcet voting-profile is given by the following 5-fold sum

$$2 \sum_{i_1=0}^{n-2} \sum_{i2=0}^{n-2-i_1} \sum_{i_3=0}^{n-2-i_1-i_2} \sum_{i4=0}^{n-2-i_1-i_2-i_3} \sum_{i_5=0}^{n-2-i_1-i_2-i_3-i_4} (2n-1)! \cdot$$

$$\frac{p_{123}^{n-1-i_2-i_3-i_5}}{(n-1-i_2-i_3-i_5)!} \cdot \frac{p_{132}^{i_2}}{i_2!} \cdot \frac{p_{213}^{i_3}}{i_3!} \cdot \frac{p_{231}^{i_2+i_4+i_5+1}}{(i_2+i_4+i_5+1)!} \cdot \frac{p_{312}^{n-1-i_1-i_2-i_4}}{(n-1-i_1-i_2-i_4)!} \cdot \frac{p_{321}^{i_1}}{i_1!} \quad .$$

This enables us to easily compute the Condorcet probability for $n \leq 50$ (i.e. up to 99 voters). Alas, while we know *for sure* that there is a recurrence, for general $p_{123}, \dots, p_{321}$, it was too complicated for our computer to discover, since it involves *symbol-crunching* with **five** free parameters.

Nevertheless, we found recurrences for two important special cases. One with two parameters, and the other one with one parameter. Before stating them, let's make a few observation about the *limiting probabilities.*

If in the three inequalities $(C12)$,$(C23)$,$(C31)$, you replace $x_{123}$ by $p_{123}$ etc., and they **all strictly hold**, or **all strictly do not hold**, then, since the expectations of $x_\pi$ is $n\,p_\pi$ for $(\pi \in S_3)$, it follows from the *Law of Large Numbers*, that the probability of a Condorcet scenario always tends to 1 (with cycles 1231 and 3213 respectively), as $n$ goes to infinity. If one of the three inequalities strictly holds but the other two stricly do not (or vice versa), then the probability goes to 0. If two of them strictly hold and one of becomes an equality, or if two of them strictly do not hold and one of becomes an equality, then the probability of a Condorcet tends to $\frac{1}{2}$.

# 4. The Border-Line Distributions

It follows that the **interesting** cases are those where all the three inequalities $(C12)$, $(C23)$, and $(C31)$, with the $x_\pi$ replaced by $p_\pi$, turn into **equalities**. These are the only probability distributions on $S_3$ where the limiting probabilities of Condorcet are not one of $\{0, \frac{1}{2}, 1\}$. This happens if and only if

$$p_{123} + p_{132} + p_{312} = p_{213} + p_{231} + p_{321} \quad ,$$

$$p_{123} + p_{213} + p_{231} = p_{132} + p_{312} + p_{321} \quad ,$$

$$p_{231} + p_{321} + p_{312} = p_{213} + p_{123} + p_{132} \quad ,$$

and of course,

$$p_{123} + p_{132} + p_{213} + p_{231} + p_{312} + p_{321} = 1 \quad .$$

It is easy to see that a parametric solution of these four equations with six unknowns is:

$$[p_{123}\,,\, p_{132}\,,\, p_{213}\,,\, p_{231}\,,\, p_{312}\,,\, p_{321}] \;=\; \left[ x\,,\, \frac{1}{2} - x - y\,,\, y\,,\, \frac{1}{2} - x - y\,,\, y\,,\, x \right] \quad ,$$

where $0 \leq x, y$ and $x + y \leq \frac{1}{2}$. We call these distributions the *borderline cases.*

We did not succeed in computing a recurrence for the Condorcet probability in the general border-line case (i.e. for general $x$ and $y$), but when $x = y$ we have the following theorem.

**Theorem 4**: Suppose that each of the $2n-1$ voters independently picks each of the rankings 123, 213, 312 and 321 with probability $x$, and each of the rankings 132 and 231 with probability $\frac{1}{2} - 2x$, where, of course, $0 \leq x \leq \frac{1}{4}$, then, the probability of a Condorcet scenario satisfies a certain fifth-order linear recurrence equation with polynomial coefficients that is too complicated to reproduce here, but can be found in the following output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oCondorcet3f.txt` .

# 5. A recurrence where the only allowed rankings are 123,231,312

We have the following theorem.

**Theorem 5**: Suppose that each of the $2n-1$ voters independently picks the ranking 123 with probability $p_{123}$, the ranking 231 with probability $p_{231}$ and the ranking 312 with probability $1 - p_{123} - p_{231}$ then, the probability of a Condorcet scenario, let's call it, $c(n)$, satisfies a certain fourth-order linear recurrence equation with polynomial coefficients that is too complicated to reproduce here, but can be found in the following output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oCondorcet3d.txt` .

**The Maple package** `Condorcet3.txt`

Everything here (and more) is implemented in the Maple package `Condorcet3.txt`, available from the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cond3.html` ,

where there are also numerous output files. The main procedures are

• `NuCo(n)`: inputs a positive integer $n$ and outputs the exact number of Condorcet voting profiles with three candidates and $2n-1$ voters.

For example, to get the number with 19999 voters, type `NuCo(10000);` .

• `PrCo(n,P)`: that inputs a positive integer $n$ and a probability distribution on $S_3$,

$$P = [p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}] \quad ,$$

and outputs the probability of a Condorcet scenario with three candidates and $2n-1$ voters. Since we were unable to find a recurrence, it does it directly, using Proposition 2. Hence can only go as far as $n = 50$ in a reasonable amount of time. For example, to get the probability (in decimals) of a Condorcet scenario with 59 voters and probability distribution $[1/8, 1/8, 1/8, 1/8, 1/8, 3/8]$, type

`evalf(PrCo(30,[1/8,1/8,1/8,1/8,1/8,3/8]);` ,

getting $0.01683856135343631855 3\ldots$.

The corresponding probability with 79 voters is gotten from

`evalf(PrCo(40,[1/8,1/8,1/8,1/8,1/8,3/8]));`, getting $0.00888551620820918\ldots$. Note that it tends to 0, as it should.

• `PrCoSp(n,p123,p231)`: it inputs a positive integer `n` and positive numbers `p123, p231` such that their sum is less than 1, and computes the Condorcet probability with $2n-1$ voters if each voter picks, independently, 123 with probability `p123`, 231, with probability `p231` and 312 with probability `1-p123-p231`. Since it uses the above-mentioned recurrence, one can go very far.

For example, to get the probability when there are 1999 voters and the probability of a 123 is $\frac{1}{4}$, the probability of a 231 is $\frac{1}{5}$, (and hence the probability of 321 is $\frac{11}{20}$, type

`evalf(PrCoSp(1000,1/4,1/5));` ,

getting $3.683198869 \cdot 10^{-6}$.

• `PrCoEq(n,x)`: it inputs a positive integer `n` and a positive number `x` (between 0 and $\frac{1}{4}$, or leave $x$ symbolic), and it outputs the probability of a Condorcet scenario with probability distribution

$$[p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}] = [x, \frac{1}{2} - 2x, x, \frac{1}{2} - 2x, x, x] \quad .$$

(Note that this is a *borderline case*). For example, typing

`evalf(PrCoEq(2000,1/10));` ,

gives 0.07718855192, that tells you that this is the probability of a Condorcet scenario (with three candidates, as always in this paper) with 3999 voters, if each voter, picks, independently, 123, 213, 312, and 321 each with probability $\frac{1}{10}$ and picks 132 and 231 each with probability $\frac{3}{10}$.

Again since we have a recurrence, we can go very far. Note that $x$ can also be *symbolic.* Typing

`plot(PrCoEq(100,x),x=0..1/4);`

would give a nice plot viewable here:

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cond3plot.html` .

In addition we have a simulation procedures `ApCo(v,P,K)`, that estimates the probability of a Condorcet scenario with a loaded die $P$ and $v$ voters, by running it $K$ times. We are happy to report that all our predictions were confirmed, within the *sampling error.*

# 6. Conclusion and Further Work

We have defined an elegant, human-generated, bijection between compositions of $n-2$ into six parts, and *Condorcet vote-count profiles* with three candidates and $2n-1$ voters, that proved that their number is $2\binom{n+3}{5}$. Next we employed this useful bijection, combined with algorithmic proof theory [PWZ] and experimental mathematics, to enumerate *Condorcet voting profiles* with three candidates, and to efficiently compute probabilities of interest.

We implemented everything in Maple, and produced lots of interesting output. Most importantly we have created two new sequences for the OEIS [S].

It would be interesting to extend our approach to other situations, for example the one discussed in [MTV], and for a number of candidates larger than three.

# 7. Acknowledgment

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