Three-Rowed CHOMP

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Abstract: A "meta" (pseudo-) algorithm is described that, for any fixed k, finds a fast ($O(\log(a))$) algorithm for playing 3-rowed Chomp, starting with the first, second, and third rows of lengths a, b, and c respectively, where $c \leq k$, but a and b are *arbitrary*.

How To Play CHOMP

David Gale's famous game of Chomp ([Ch]) starts out with an M by N chocolate bar, in which the leftmost-topmost square is poisonous. Players take turns picking squares. In his or her (or its) turn, a player must pick one of the remaining squares, and eat it along with all the squares that are "to its right and below it". Using matrix-notation with the poisonous square being entry $(1, 1)$, and the initial position consisting of the whole bar $\{(i, j)|1 \leq i \leq M, 1 \leq j \leq N\}$, then picking the square (i_0, j_0) means that one has to eat all the remaining squares (i, j) for which *both* $i \geq i_0$ and $j \geq j_0$ hold. The player that eats the poisonous (leftmost-topmost) square loses. Of course picking $(1, 1)$ kills you, so a non-suicidal player will not play that move unless it is forced to.

For example, if $M = 4$ and $N = 3$, then the initial game-position is

$$\begin{matrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \end{matrix} \quad .$$

The first player may choose to play $(4, 3)$, in which case the game-position becomes

$$\begin{matrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & \end{matrix} \quad ,$$

or he may choose to play $(2, 2)$, which shrinks the chocolate bar to

$$\begin{matrix} X & X & X \\ X & & \\ X & & \\ X & & \end{matrix} \quad ,$$

and so on.

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The Two Cultures of Mathematics: Existential and Constructive

Nineteenth century mathematicians were honest. When they tried to prove that something existed, they tried to find it, *explicitly.* Then came Hilbert along, and turned math into *theology* by "proving" existence by using the *"Law" of the Excluded Middle.* Of course, 20th-century mathematicians were forced to cheat, because they ran out of "computing time" and "memory allocation" in their tiny PC between their shoulders.

Now that we have much more powerful silicon brains to aid us, we can afford again the luxury of being honest, and actually *finding* the desired objects, at least whenever it is theoretically possible.

Of course, in (finite) combinatorics, there is no philosophical objection to using indirect arguments, but it is still much more satisfying to *exhibit* the object rather than just proving its existence.

One of the most gorgeous existence proofs of all times is the following gem of David Gale ([Ch]).

The Chomp Existence Theorem: In a Chomp game that starts with an M by N chocolate bar (with $MN > 1$), there *exists* a winning move for the first player.

David Gale's One-Line Proof: Consider the minimal move of only removing the square at the bottom-rightmost corner, (M, N). If it is a winning move, then it is a winning move. Otherwise, it is a losing move. But in the latter case, this means that your opponent has a good counter-move. It is immediate that any position reachable from the new position (with $MN - 1$ squares) is also reachable from the initial position (with MN squares). Hence the first player could have gotten to that position right away.

This famous (but not famous enough!) proof can be used in practice as follows. I first play with Kasparov, pretending that I am Deep Blue, and make the above trial move of removing the bottom-right corner. If he resigns, then it is a good move, otherwise, he would make a good counter-move, that I can use when I play with you. This idea is called 'strategy stealing' in [WW].

How to Find the Winning Move?

Of course, there *exists* an algorithm for finding the winning move, and more generally of deciding whether any given Chomp position is winning or losing, and finding a winning move in the former case. This is true for all *combinatorial games* where the number of possible positions is finite, and the game graph is acyclic. This is part of the lovely theory of *combinatorial games.* Readers not familiar with this beautiful theory are strongly advised to read Aviezri Fraenkel's lively and very lucid introduction [ST].

First one "draws" the *game graph* of the game. This is a directed graph whose vertices are all possible "game positions", and there is a directed edge between the vertex corresponding to position P_1 and the vertex corresponding to position P_2 if P_2 is reachable from P_1 in *one* legal move.

Having 'drawn' that graph, you label all the sinks (vertices with outdegree 0) by $\mathcal{P}$ (for 'Previous player wins'), because if it is your turn to move and you are there, then you lost, because you can't go anywhere. Next label all vertices that have at least one edge leading to a $\mathcal{P}$-vertex, by $\mathcal{N}$ (for "Next player wins"). Then, if you see a vertex all of whose edges lead to $\mathcal{N}$-labelled vertices, label it $\mathcal{P}$. Continue to alternately label the vertices $\mathcal{N}$ and $\mathcal{P}$ until you finish.

If the directed graph of the game is finite and has no cycles, then it is easy to see that the above procedure always terminates.

Now let's apply this to Chomp. How many possible positions are there? The intermediate positions in an $M \times N$ Chomp are integer-partitions (non-increasing sequences of positive integers) $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, with $\lambda_1 \leq N$ and $r \leq M$. It is well known and easy to see that their number is $\binom{M+N}{N}$. So for example when $M = O(N)$, we have an exponential size graph, and hence an exponential-time and exponential-memory algorithm for labelling it (and doubly-exponential in bit size!).

When $M = N$, there is a trivial winning move. Choose to chomp at the square at location $(2, 2)$, leaving a symmetric L-shaped shape, and then use the *copy-cat* strategy of aping your opponent's move to the other arm of the L-shape.

Another trivial special case is when we only have 2 rows. We have

Proposition 0: A 2-rowed Chomp position is $\mathcal{P}$ iff it is of the form $(a, a-1)$ $(a \geq 1)$, i.e. where the top row has one more square than the bottom row. If (a, b) $(a \geq b \geq 0)$ is a Chomp position that is winning (i.e. $b \neq a-1$), then if $a - b \geq 2$, a winning move is to go to position $(b+1, b)$, and if $a - b = 0$, to go to position $(b, b-1)$.

Proof: We really have two statements here:

$P_1(a)$: $(a, a-1)$ is a $\mathcal{P}$ position (for all $a \geq 1$).

$P_2(b)$: (a, b) is an $\mathcal{N}$ position for every a such that $a \geq b \geq 0$, and $a - b \neq 1$. Furthermore the winning move is $(a, a-1)$ if $a = b$ and $(b+1, b)$ if $a - b \geq 2$.

The proof is by joint induction. $P_1(1)$ is trivially true. The positions reachable from $(a, a-1)$ are $(a-1, a-1)$, and $(a, a-2)$,$(a, a-3), \ldots, (a, 0)$, each of which are $\mathcal{N}$ positions because of $P_2(b)$, for $b < a$. Hence $\mathcal{P}_2(b)$ for $b < a$ imply $P_1(a)$.

But also $P_2(b)$ is implied by $P_1(b-1)$, if $a = b$, and by $P_1(b)$ if $a - b = 2$.

So we have that $P_1(b-1)$ and $P_1(b)$ imply $P_2(b)$, and $P_2(i), i = 1 \ldots b-1$ imply $P_1(b)$. Since $P_1(1)$ is true, this completes the induction.

What is the Computational Complexity of 2-Rowed Chomp?

First note that when we fix the number of rows and play $k \times n$-Chomp, and only let the number of columns, n, get larger, then the naive algorithm is no longer exponential in n, but rather is

$O(n^k)$. But in order to *represent* a position in k-rowed Chomp, we only need $O(\log n)$ bits, hence the naive algorithm is still exponential in the size of the input for any fixed number of rows, and hence doubly-exponential for general Chomp.

On the other hand, the algorithm of Prop. 0, for 2-rowed Chomp, is clearly linear in the bit-size of the input, both for deciding whether it is a $\mathcal{P}$ or $\mathcal{N}$ position, and for finding a winning move in the latter case.

IMPOSSIBLE DREAM: Find a polynomial-time algorithm (in the bit-size of the input) for Chomp.

In particular, such an algorithm would quickly (i.e. in time polynomial in $\log n$ and k) find the winning first move in a $k \times n$ Chomp, whose *existence* is guaranteed by Gale's clever strategy-stealing argument.

Since the above dream is impossible, we have to find more realistic dreams. We should not be like the fictional Uncle Petros from Apostolos Doxiadis's novel, who 'wasted' his life trying to prove the Goldbach Conjecture, and like the real Paul Cohen, who is rumored to have 'wasted' the second half of his life trying to prove the Riemann Hypothesis.

So let's be more modest and try to find a fast algorithm (in $\log n$) for k-rowed Chomp, where k is fixed. Even this is probably impossible. So let's be even more modest and try to do it for $k = 3$, granted that we already know how to do it for $k = 2$. Even this seems to be impossible, at least for me. So what is a poor mathematician to do? The case $k = 2$ is known and trivial, while the case $k = 3$ is (probably) impossible. This is just one instance of the human mathematician's predicament of walking the

TIGHTROPE BETWEEN the TRIVIAL and the IMPOSSIBLE

One may define the *cutting edge of research* as that x for which doing $x - \epsilon$ is utterly trivial while doing $x + \epsilon$ is impossible.

It is not unlike designing a math exam for my calculus students. If I give them the kind of tests that I used to take, and even much easier ones, they will all flunk. On the other hand, if I'll ask them to find $2+2$, $2+3$, and problems like that, and allow calculators, most of them will get an A. It is non-trivial to design a test that is of "just right" level of difficulty to produce a bell-shaped curve.

This 'trivial for $k = 2$, impossible for $k = 3$' curse brings to mind the "Three Body Problem". One way out of the impossibility dead-end is to abandon the *quantitative* and go to the *qualitative*, as did Poincaré. This lead to another cultural divide.

The Two Cultures of Mathematics: Quantitative and Qualitative.

But proving existence, uniqueness, and ergodicity will never get us to the moon, nor will it predict the planetary orbits for the next ten thousand years. For that, mathematicians took advantage of the fact that planets are much lighter than the Sun, and used *perturbation expansion*, starting with the solution of the 2-body problem. This leads to

The Two Cultures of Mathematics: Exact and Approximate.

The approximate culture gave rise to numerical analysis and its theoretical parent, approximation theory. It is approximate mathematics that is the most useful in the sciences. The whole edifice of Feynman diagrams was designed to find 'series expansions' that go from a trivial ϕ^2 (Gaussian) functional integral to an impossible ϕ^3-and-beyond functional integral. Its practical effectiveness and *amazing agreement with experiment* is due to the lucky coincidence(?) that the fine structure constant is fairly small (roughly 1/137). The default in physics is 'perturbative', and whenever a 'non-perturbative' answer is found, like in Nati Seiberg's astounding tour-de-force that lead to the Seiberg-Witten invariants and to much more, it is a cause of celebration in physics (and in this case also in math, because of its far-reaching implications in topology).

Speaking of Quantum Field Theory, most of it is non-rigorous, and this brings to mind another cultural divide, this time involving the broader community of people who use mathematics, as opposed to full-time mathematicians.

The Two Cultures of Doing Mathematics: Rigorous and Non-Rigorous.

According to Pierre Cartier's charming and fascinating article [Ma] professional mathematicians should pay more attention to the latter, since ([Ma], p.1) *"There is another way of doing mathematics, equally successful, and the two methods should supplement each other and not fight"*. Yet another way, is to compromise and do *semi-rigorous math* as I proposed in my celebrated *manifesto* [SR].

Back to Chomp

But the term 'approximation' is not appropriate in this context, since everything is discrete. A position is either $\mathcal{P}$ or $\mathcal{N}$. *Perturbation* is more pertinent. An arbitrary 3-row Chomp position can be described as (a, b, c) where a,b, c are the lengths of the top, middle, and bottom rows respectively, and of course $a \geq b \geq c \geq 0$. When $c = 0$, we are back to 2-row Chomp, so the next thing to try is to characterize the $\mathcal{P}$ positions when $c = 1$. Let's try and do it. We know that $(b + 1, b, 1)$ is $\mathcal{N}$, since chomping the bottom cell (the sole cell of the bottom row), leads to the $\mathcal{P}$ position $(b+1, b, 0)$. We also know that $(1, 1, 1)$ is $\mathcal{N}$, since the $\mathcal{P}$ position $(1, 0, 0)$ is reachable from it. We also know that $(a, 0, 0)$, for $a > 1$, is $\mathcal{N}$, since we can get to $(1, 0, 0)$ from it. From the 2-rowed case we know that $(a, a, 0)$, and $(a, b, 0)$ with $a - b \geq 2$ are $\mathcal{N}$. What are the 'smallest' positions with $c \leq 1$, that are $\mathcal{P}$?. We see that all the four-cell positions: $(4, 0, 0), (3, 1, 0), (2, 2, 0), (2, 1, 1)$ are $\mathcal{N}$. Amongst the five-cell positions: $(5, 0, 0), (4, 1, 0)$ are $\mathcal{N}$, since the former can be answered by $(1, 0, 0)$ and the latter by $(2, 1, 0)$. We already know, from the 2-rowed case, that (3,2,0) is $\mathcal{P}$, and now we see that (2,2,1),(3,1,1) are both $\mathcal{P}$, since all the positions reachable from them are $\mathcal{N}$ (do it!).

So, now we have two new members of the $\mathcal{P}$ club: $(2,2,1)$, and $(3,1,1)$. But this two factoids entail an infinite number of new memberships in the $\mathcal{N}$ club!, namely $(2+\alpha, 2+\beta, 1)$ for all $\alpha \geq \beta \geq 1$, as well as $(2+\alpha, 2, 1)$ for $\alpha \geq 1$. Also, from the fact that $(3,1,1)$ is $\mathcal{P}$ we get that $(3,2,1), (3,3,1)$ are $\mathcal{N}$, as well as $(3+\alpha, 1, 1)$ for $\alpha \geq 1$. Now it is easy to see (do it!) that these $\mathcal{N}$ positions exhaust all the possible positions with more than 5 cells and $c \leq 1$. Hence:

Proposition 1: The only $\mathcal{P}$ positions (a,b,c), with $c = 1$, are $(2,2,1)$ and $(3,1,1)$. Every $\mathcal{N}$ position with $c = 1$, and at least 6 cells, is (at least) in one of the forms: $(3,2,1)$;$(3,3,1)$;$(4+\alpha,1,1)(\alpha \geq 0)$; $(2+\alpha, 2+\beta, 1)(\alpha \geq \beta \geq 0, \alpha+\beta > 0)$ where $(\alpha \geq \beta \geq 0)$, and the winning moves are, respectively:

$$(3,2,1) \rightarrow (3,1,1) \quad ,$$

$$(3,3,1) \rightarrow (3,1,1) \quad ,$$

$$(4+\alpha,1,1) \rightarrow (3,1,1) \quad ,$$

$$(2+\alpha,2+\beta,1) \rightarrow (2,2,1) \quad .$$

Note that there turned out to be only 2 members of $\mathcal{P}$ when $c = 1$, so we did not need induction. On the other hand when $c = 0$ (the 2-rowed case), there were infinitely many members, so we had to resort to induction.

Now we are ready to graduate to the case $c = 2$. From the $\mathcal{P}$ positions for $c = 0$ we know that $(b+1, b, 2)$ all belong to $\mathcal{N}$ (you chomp-off the last row), and from $(3,1,1) \in \mathcal{P}$ we know that $(3,2,2) \in \mathcal{N}$, and from $(2,2,1) \in \mathcal{P}$ we know that $(2,2,2) \in N$. Hence the smallest (according to the number of cells) $\mathcal{P}$ position is $(4,2,2)$, since all the possible moves lead to $\mathcal{N}$ positions. Now this fact implies that $(4,3,2), (4,4,2)$ are $\mathcal{N}$ positions. The smallest $\mathcal{P}$ position is $(5,3,2)$, which immediately implies that $(5,4,2) \in \mathcal{N}$ and $(5+\alpha, 3, 2) \in \mathcal{N}$, for $\alpha \geq 1$. The smallest position that is not covered by the above is $(6,4,2)$, and we can directly verify that all the possible moves from $(6,4,2)$ lead to $\mathcal{N}$ positions. So $(6,4,2) \in \mathcal{P}$. Continuing, we get that also $(7,5,3) \in \mathcal{N}$, and we humans would be soon lead to conjecture the following

Proposition 2: A Chomp position $(a,b,2)$ is $\mathcal{P}$ iff $a - b = 2$.

Now this is a *genuine* proposition, since it contains infinitely many statements: $(4,2,2) \in \mathcal{P}$, $(5,3,2) \in \mathcal{P}$, $(6,4,2) \in \mathcal{P}, \ldots$. Nevertheless, we humans can easily prove it by induction like we did prove Proposition 0 for $c = 0$ (the two-rowed case). Why won't YOU do it RIGHT NOW?!

We can continue in this vein, but I doubt that any human can go beyond $c = 10$, since the number of cases grows larger and larger, as we will see below, once we let Shalosh B. Ekhad take over. One hope would be that there would emerge a *pattern* valid for arbitrary c, that a human would be able to prove by induction or otherwise. However this miracle is probably as unlikely as a closed-form solution to the Three Body Problem. Neither I, nor Shalosh, were able to detect such a pattern, and while Shalosh succeeded, using the Maple package Chomp3Rows, to be described soon, to first conjecture, and then (all by itself!) prove Proposition c_0, for $c_0 \leq 115$, characterizing the $\mathcal{P}$

positions of the form (a, b, c_0), no discernible *global* pattern emerged (as a function of c_0), at least not to us.

So Why is it Interesting?

Gregory Chaitin, standing on the shoulders of Kurt Gödel and Alan Turing, taught us that most results are either undecidable or uncompressible. Whenever there is a pattern, it means that the result is compressible, hence was trivial all-along, we just did not know it. So we humans are destined to only prove trivial results, since any result, once proved, is trivial, for the very reason that *we*, low-tech humans, were able to prove it. The only way to transcend this triviality predicament, and to be able to prove semi-trivial results, is to ask the computer to help us (see [Op36][Op37]).

In other words, we have this powerful hammer, called the computer, that can hammer so many different nails. Of course, we should not waste our time hammering screws, but when we see a screw, rather than look for a screwdriver, or try to use our mighty hammer to awkwardly hammer it, just leave it alone! There are so many challenging nails around, where our hammer can be used to advantage, that we should keep a lookout for them, and try to nail them, thereby hopefully improving our hammering skills. It would be premature to try and find a computer-proof of the Riemann Hypothesis, so let's try first to teach the computer how to do research in, say, 3-rowed Chomp. Whenever, we have a class of human theorems where the arguments repeat themselves, it is a good candidate for teaching a computer. So the main justification for developing the package *Chomp3Rows* was as an *étude* in computer-generated research. As we'll get better and better at it, very soon we will be able to tackle the Riemann Hypothesis and other biggies.

I call people like me, 'mathematical engineers', and I am sure that this culture of mathematical (software) engineering, will soon be the mainstream one, making the following dichotomy obsolete.

The Two Cultures of Mathematics: Problem Solvers and Theory Builders.

W.T. Gowers [TC], in a fascinating, though somewhat defensive, article, described these two cultures, lamenting that Theory Builders are still the topdogs, and problem-solvers the underdogs. Luckily, this is no longer true! In fact, he himself, a problem-solver by his own avowal, is a living proof of the respectability of problem-solvers, having won the Fields medal.

Since computers are more suited for problem-solving than theory-building (at least for some time to come), I believe that problem-solving will become more and more important. But, with a twist! It would be a total waste of time for a human to solve a problem directly. The mainstream activity of late 21st and 22nd century mathematics would be meta-problem-solving: programming the computer to solve problems, that we humans would never have a chance to prove by ourselves. So Gowers's two breeds of mathematicians will both die out and give place to the

Mainsteam Culture of Future Mathematics: Programming Computers to do Math.