How Sister Celine Fasenmyer and Dick Duffin Shaped my Mathematical Personality

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Two of my great heroes passed away last fall. First Richard James Duffin, at the end of October, and then Sister Celine Fasenmyer, at the end of December. Both of them lived to a ripe old age (87 and 90 resp.).

Apart from their longevity, they seem to have nothing in common. Dick was recognized as a leading applied mathematician, was a member of the National Academy of Science, had many brilliant Ph.D. students, (at least one of whom (Raul Bott) who surpassed him in fame, and at least another (Hans Weinberger) of comparable fame), had over two-hundred publications, etc. etc. On the other hand, Sister Celine Fasenmyer was an obscure college professor, who did not publish anything beyond her thesis work, and of course never had any Ph.D. students.

A closer look reveals that they are not as different as they appear to be at first sight. First, I believe that they were both grossly under-rated. Sister Celine's greatness only started to emerge with the WZ theory, and her story is told in the classic $A{=}B$. I am sure that the future will prove her even greater.

Dick Duffin's creative genius, while partly recognized, sure did not get the full recognition it deserved: a place among the top 20 mathematicians of our century. The main reason that Dick did not get his proper recognition-due was that he loved Mathematics more than he liked being famous. In the pompous era of Bourbakism, he preferred to have fun and solve problems. Often these problems lead to new theories, but he never made a big deal of it, continuing to the next problem. His problem-solving love can be gleaned from his lovely Bulletin article (*Bull. Amer. Math. Soc.* **80***(1974), 1053-1070)*, extended in 'Constructive Approaches to Mathematical Models' (Proc., in honor of R.J. Duffin), C.V. Coffman, and G.J. Fix, eds., Academic Press, 1979,pp 3-32.

Duffin's Bulletin paper looks like, and in some sense is, a disjointed collection of 'math-bites'. However, each of these bites is a magnificent contribution, and the sum of the parts is very considerable. A more global view would show that the whole is even more magnificent than the sum of the parts, and I hope that in the future, his deep insight, combined with his down-to-earth unpretentious style of doing mathematics, will be emulated. I believe that this paper should be required reading to any budding graduate student in mathematics, both pure and applied.

But, the most important thing that Sister Celine and Dick Duffin share, is a place in my personal Pantheon of heroes, who shaped my mathematical personality.

According to Sigmund Freud, an individual's personality is formed in the first five years of his or her life. The mathematical analog of the formative years is the period of one's graduate studies. It was then that I encountered, quite by accident, both Dick Duffin and Sister Celine.

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My advisor, Harry Dym, gave me a problem in complex analysis. But, as hard as I tried, all the approaches that he suggested did not seem to work. After a year and a half, I was still stumped. Then I had my own idea: to solve the problem by doing the discrete analog first. I went to the library, and started browsing, completely at random, at current and old math journals. (Of course MathSciNet did not exist back in 1974, and I was not systematic enough to look at Math Reviews.) I then bumped into an article by Duffin and his student Charles Duris, that had a reference to Dick's seminal paper on discrete analytic functions, that appeared in the Duke Journal, in 1956.

I immediately fell in love with Discrete Analytic Functions. I also realized that I was not born to be a continuous analyst, but a discrete one. Of course, knowing discrete analytic functions did not help me a bit with the original thesis problem, but I did not care, and I duly abandoned it, in favor of proving discrete analogs of any theorem in complex analysis I could think of.

Neither Harry, nor the other professors, in the then predominantly analysis math department at the Weizmann Institute, wholly approved of this change of direction. To them it seemed a futile exercise in 'formal analogs', and Yakar Kannai even pleaded with the great function-theorist, Larry Zalcman, who was visiting the department, to 'talk me out of this nonsense'. In a sense they were right. Doing Discrete Analytic Functions, was not the most optimal way to get an academic job.

Later, I also came to realize that as a theory *per se*, Discrete Analytic Functions would never have the rich texture of its continuous namesake. But, studying Dick Duffin's and his students' papers lead to something far more important. The *realization* that the Continuous and Discrete should be studied side by side. Even more significantly, the *operator notation* that Duffin widely used.

It was Duffin who introduced me, through his papers, to this very powerful notation. A *discrete analytic function* is a complex-valued function on the discrete two-dimensional lattice Z^2 such that

$$\frac{f(m+1,n+1)-f(m,n)}{1+i} = \frac{f(m,n+1)-f(m+1,n)}{i-1} \quad .$$

In other words the 'directional (discrete) derivative' in the NE diagonal equals that in the NW diagonal, at each lattice point. By cross multiplying,

$$f(m+1,n+1) - if(m,n+1) + if(m+1,n) - f(m,n) \equiv 0 \quad .$$

Duffin defined $Xf(m,n) := f(m+1,n)$ and $Yf(m,n) := f(m,n+1)$, and rewrote the above as:

$$(XY - iY + iX - 1)f(m,n) \equiv 0 \quad .$$

In other words f is *discrete analytic* if it is annihilated by the "discrete Cauchy-Riemann" *partial difference operator* $XY - iY + iX - 1$.

Thinking of operators, it was natural to think of them as *polynomials*. While the operator above is *constant coefficients*, one can conceive of linear difference operators with *variable coefficients*. A few months later, also quite by accident (or so it seemed), I leafed through Rainville's book on Special Functions (reprinted by Chelsea, 1971) and saw the chapter on Sister Celine's technique. I realized that with Duffin's operator-notation, things become much more transparent and generalizable, and that Sister Celine's technique is essentially a non-commutative analog of classical elimination methods (notably Sylvester's and Bezout's) from commutative algebra.

Thus was born what later became WZ theory. Herb Wilf added the crucial notions of *WZ pair*, *certificate* and *duality*, which I would never have been able to conceive precisely because my random walk was via Dick and Sister Celine, and Herb's random walk came from elsewhere. □