Boolean Function Analogs of Covering Systems

Anthony ZALESKI and Doron ZEILBERGER

Prolog: Primes Numbers are (often) Red Herrings

In the September 2017 of Jean-Paul Delahay's wonderful Logique & Calcul column ([D]), the following variation on one of the numerous *Ant Alice* puzzles in Peter Winkler's wonderful book ([W]).

On places nine beetles on a circle, the distances between two consecutive beetles, measured in meters, along the circle are the first nine primes $2, 3, 5, 7, 11, 13, 17, 19, 23$*. The order is arbitrary, but each of the distances occurs exactly once.*

At the starting time, each beetle decides, randomly, whether it is going to walk, at a speed of 1 *meter per minute, clockwise or counterclockwise. When two beetels collide, they immediately turn-around. We assume that the size of the beetles are negligible. After* 50 *minutes, after many collisions and such "U-turns" of the nine beetles, one notes their position and the distances that separate them. These nine distances are still the same nine primes! How can you explain this arithmetical miracle?"*

Please think about it for a few minutes and see whether you can explain this miracle (if you have not seen it before).

The beautiful solution is as follows. Let each beetle carry a flag, and whenever they collide, they exchange their flags, like in a relay race. It folows that each of the flags moves in its initial direction at a speed of 1 meter per minute. Since the lenght of the "track" is

$$2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 = 100 \quad .$$

After 50 minutes, each flag moved exactly to the opposite point on the circle, hence the relative distances between consecutice flags (and hence consecutive beetles) remain the **same**. QED.

Delahaye made two changes to Winkler's original puzzle. First he changed *ants* to *beetles*, but **very sneakily** made the initial distances primes! The solution shows that if the initial distances are $a_1, \dots, a_k$, then the distances are preserved after $(a_1 + \dots + a_k)/2$ minutes (and of course, any multiple of it). So choosing **primes** was a huge *red herring*, that made this delightful brain-teaser so much harder.

We claim that primes are (very possibly) also red herrings in much more "serious" problems, even, who knows?, the Riemann Hypothesis, and the Goldbach conjecture.

References

[D] Jean-Paul Delayahe, *Cinq énigmes pour la rentrée*, Logique & Calcul, Pour La Science, **No. 479**, Septembre 2017, 80-85.

[W] Peter Winkler, "*Mathematical Mind-Benders*", A.K. Peters, 2007.

Anthony Zaleski, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `az202 at math dot rutgers dot edu` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .