Bijections for an identity of Young Tableaux

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Recall that *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly-decreasing sequence of positive integers, and if $n = \lambda_1 + \cdots + \lambda_k$ we say that λ is a partition of n and write $\lambda \vdash n$ and $l(\lambda) = k$. For example $\lambda(3,3,2,2)$ is a partition of 10 and $l(\lambda) = 4$. A convenient way to represent a partition λ is via its *Ferrers* diagram, that consists of $l(\lambda)$ left-justified lines of dots, such that the i-th row has λ_i dots. If you replace the dots by empty boxes, you would get what is called a *Young* diagram. Finally recall that a *standard Young Tableau* (SYT) of shape $\lambda \vdash n$ is any way of placing the integers $\{1, 2, \ldots, n\}$ into the empty boxes of the Young diagram in such a way that all the rows and all the columns are increasing. For example $[[1,2,4,6],[3,5,7],[8,9]]$ is a SYT of shape $(4,3,2)$. Let $H(k,\ell;n) = \{\lambda \vdash n \mid \lambda_{k+1} \leq \ell\}$ denote the partitions of n in the (k,ℓ) hook. For example, $H(k,0;n)$ are the partitions of $\lambda \vdash n$ with $\ell(\lambda) \leq k$. Let f^λ denote the number of standard Young tableaux (SYT) of shape λ. One then observes the following intriguing identity:

$$\sum_{\mu \in H(1,1;n+1)} (f^\mu)^2 = \sum_{\lambda \in H(2,0;2n)} f^\lambda.$$

We give this identity a bijective proof by showing that both

$$\sum_{\mu \in H(1,1;n+1)} (f^\mu)^2, \tag{1}$$

$$\sum_{\lambda \in H(2,0;2n)} f^\lambda, \tag{2}$$

can be mapped *bijectively* to the set of row-increasing matrices of shape (n,n), and hence, by composing, to each other. We describe these bijections.

The input for both (1) and (2) is a $2 \times n$ matrix of integers

$a_1 \ldots a_n$
$b_1 \ldots b_n$

such that $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots b_n\} = \{1, 2, \ldots, 2n\}$,

$a_1 < a_2 < \ldots < a_n$ and
$b_1 < b_2 < \ldots < b_n$.

Description of the bijection for (1)

Here the output is: Two standard tableaux of the same $(1,1)$-hook shape. Let $|\{a_1, \ldots, a_n\} \cap \{1, \ldots, n\}| = k$, so $\{a_1, \ldots, a_n\} \cap \{1, \ldots, n\} = \{a_{i_1} < \cdots < a_{i_k}\}$. Form now a SYT in the $(1,1)$ hook as follows:

Its (first) row is

$$1, \quad a_{i_1}+1, \quad \cdots \ , a_{i_k}+1;$$

its (first) column is made of the remaining integers

$$\{1, \dots n+1\} \setminus \{\ 1,\ a_{i_1}+1, \quad \cdots \ , a_{i_k}+1\} = \{a'_{j_1}, \dots, a'_{j_{n-k}}\}$$

in increasing order. This gives the first SYT – of $(1,1)$-hook shape $(k+1, 1^{n-k})$.

It follows that

$$|\{a_1, \dots, a_n\} \cap \{n+1, \dots, 2n\}| = n-k,$$

so denote

$$\{a_1, \dots, a_n\} \cap \{n+1, \dots, 2n\} = \{b_{t_1} < \cdots < b_{t_{n-k}}\}.$$

Since $n+1 \le b_{t_1}$, hence $2 \le b_{t_1} - (n-1)$. Form the numbers

$$1 < b_{t_1} - (n-1) < \cdots < b_{t_{n-k}} - (n-1)$$

and place them, in that (increasing) order, in the column of the second (1,1)-hook shape tableau, and the complement integers

$$\{1, \dots, n+1\} \setminus \{1, b_{t_1} - (n-1), \dots, b_{t_{n-k}} - (n-1)\}$$

in increasing order, in the row (after the corner 1). This gives the second SYT – again of shape $(k+1, 1^{n-k})$. This map is clearly a bijection.

Description of the bijection for (2)

Here the output is a SYT whose shape is a $\le$ 2-rowed partition of $2n$ If for all i $a_i < b_i$, then it is a SYT, and do nothing (the output is the input).

Otherwise, let i be the smallest index such that $a_i > b_i$. Replace the above array by

$$\begin{array}{lllllll} b_1 \ \dots \ b_{i-1} & b_i & a_i & a_{i+1} & \dots & a_n \\ a_1 \ \dots \ a_{i-1} & b_{i+1} & b_{i+2} & b_{i+3} & \dots & b_n \end{array}$$

If this is a SYT, then stop. Otherwise continue: Typically we arrive at a two rows array

$$\begin{array}{l} c_1, \ \dots \ c_s \ \dots \ c_r \\ d_1 \ \dots \ d_s \end{array}$$

with $s \le r, \quad r+s = 2n$, with $c_1 < \cdots < c_r$ and with $d_1 < \cdots < d_s$. If $c_j < d_j, \ \ j = 1, \dots, s$ then this array is SYT and we are done. Otherwise let i be the smallest index such that $c_i > d_i$, then replace the above array by

$$\begin{array}{lllllll} d_1 \ \dots \ d_{i-1} & d_i & c_i & c_{i+1} \ \dots c_{s-1} & \dots & c_r \\ c_1 \ \dots \ c_{i-1} & d_{i+1} & d_{i+2} & d_{i+3} \ \dots \ d_s \end{array}$$

Continue until a SYT is reached: thus the process stops at a SYT of shape $\lambda \in H(k, 0; 2n)$.

Remark. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition, $\lambda_1 \geq \lambda_2 \geq \ldots$ and denote $\lambda^{+1} = (\lambda_1 + 1, \lambda_2, \lambda_3 \ldots)$. The same bijections can be applied to similar arrays of shape $(n+1, n)$, yielding a bijective proof for the SYT identity

$$\sum_{\mu \in H(2,0;2n+1)} f^{\mu} = \sum_{\lambda \in H(1,1;n+2)} f^{\lambda} \cdot f^{\lambda^{+1}}.$$

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