Tweaking the Beukers Integrals In Search of More Miraculous Irrationality Proofs À La Apéry

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In honor of our irrational guru Wadim Zudilin, on his $\lfloor 50\,\zeta(5) \rfloor$-th birthday

[Actual] Historical Introduction: How Beukers' Proofs Were ACTUALLY found

Hilbert's 0-th problem

Before David Hilbert [H] stated his famous 23 problems, he mentioned two problems that he probably believed to be yet much harder, and indeed, are still wide open today. One of them was to prove that there are infinitely many prime numbers of the form $2^n + 1$, and the other one was to prove that the Euler-Mascheroni constant is irrational.

Two paragraphs later he stated his optimistic belief that "in mathematics there is no *ignorabimus*."

As we all know, he was proven wrong by Gödel and Turing *in general*, but even for such concrete problems, like the irrationality of a specific, natural, constant, like the Euler-Mascheroni constant (that may be defined in terms of the **definite** integral $-\int_0^\infty e^{-x} \log x \, dx$) , that is most probably **decidable** in the logical sense, (i.e. there *probably* **exists** a (rigorous) proof), we *lowly humans* did not yet find it, (and may never will!).

While the Euler-Mascheroni constant (and any other, natural, explicitly-defined, constant that is not obviously rational) is **surely** irrational, in the everyday sense of the word *sure* (like death and taxes), giving a *proof*, in the **mathematical** sense of 'proof' is a different matter. While e was proved irrational a long time ago (trivial exercise), and π was proved irrational by Lambert around 1750, we have no clue how to prove that $e + \pi$ is irrational. Ditto for $e \cdot \pi$. **Exercise**: Prove that *at least* one of them is irrational.

Apéry's Miracle

As Lindemann first proved in 1882, the number π is more than *just* irrational, it is *transcendental*, hence it follows that $\zeta(n)$ is irrational for all even arguments, since Euler proved that $\zeta(2n)$ is a multiple of π^{2n} by a rational number. But proving that $\zeta(3)$, $\zeta(5)$, ... are irrational remained wide open.

Since such problems are so hard, it was breaking news, back in 1978, when 64-years-old Roger Apéry announced and sketched a proof that $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational. This was beautifully narrated in a classic expository paper by Alf van der Poorten [vdP], aided by details filled-in by Henri Cohen and Don Zagier. While beautiful in our eyes, most people found the proof ad-hoc and too complicated, and they did not like the heavy reliance on *recurrence relations*.

To those people, who found Apéry's original proof too *magical*, ad-hoc, and *computational*, another proof, by a 24-year-old PhD student by the name of Frits Beukers [B] was a breath of fresh air. It was a marvelous *gem* in human-generated mathematics, and could be easily followed by a first-year student, using *partial fractions* and very easy estimates of a certain triple integral, namely

$$\int_0^1 \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y)z(1-z))^n}{(1-z+xyz)^{n+1}} \, dx \, dy \, dz \quad .$$

The general approach of Apéry of finding concrete sequences of integers a_n, b_n such that

$$|\zeta(3) - \frac{a_n}{b_n}| < \frac{CONST}{b_n^{1+\delta}} \quad ,$$

(see below) for a **positive** δ was still followed, but the details were much more palatable and *elegant* to the average *mathematician in the street*.

As a warmup, Beukers, like Apéry before him, gave a new proof of the already proved fact that $\zeta(2) = \frac{\pi^2}{6}$ is irrational, using the double integral

$$\int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \, dx \, dy \quad .$$

Ironically, we will follow Beukers' lead, but heavily using **recurrence relations**, that will be the **engine** of our approach. Thus we will abandon the original *raison d'être* of Beukers' proof of **getting rid of recurrences**, and bring them back with a **vengeance**.

[Alternative World] Historical Introduction: How Beukers's Proofs Could (and Should!) have been Discovered

Once upon a time, there was a precocious teenager, who was also a *computer whiz*, let's call him/her/it/they *Alex*. Alex just got a new laptop that had `Maple`, as a birthday present.

Alex typed, *for no particular reason*,

```
int(int(1/(1-x*y),x=0..1),y=0..1);
```

and immediately got the answer: $\frac{\pi^2}{6}$. Then Alex was wondering about the sequence

$$I(n) := \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}} \, dx \, dy \quad .$$

(why not, isn't it a natural thing to try out for a curious teenager?), and typed

```
I1:=n->int(int(1/(1-x*y)*(x*(1-x)*y*(1-y)/(1-x*y))**n,x=0..1),y=0..1);
```

(`I` is reserved in Maple for $\sqrt{-1}$, so Alex needed to use `I1`),

and looked at the first ten values by typing:

`L:=[seq(I1(i),i=1..10)];` ,

getting after a few seconds

$$[5-\frac{\pi^2}{2}, -\frac{125}{4}+\frac{19\,\pi^2}{6}, \frac{8705}{36}-\frac{49\,\pi^2}{2}, -\frac{32925}{16}+\frac{417\,\pi^2}{2},$$
$$\frac{13327519}{720}-\frac{3751\,\pi^2}{2}, -\frac{124308457}{720}+\frac{104959\,\pi^2}{6},$$
$$\frac{19427741063}{11760}-\frac{334769\,\pi^2}{2}, -\frac{2273486234953}{141120}+\frac{9793891\,\pi^2}{6},$$
$$\frac{202482451324891}{1270080}-\frac{32306251\,\pi^2}{2}, -\frac{2758128511985}{1728}+\frac{323445423\,\pi^2}{2}] \quad .$$

Alex immediately noticed that, at least for $n \leq 10$,

$$I(n) = a_n - b_n \frac{\pi^2}{6} \quad ,$$

for some **integers** b_n and some **rational numbers** a_n. By taking `evalf(L)`, Alex also noticed that $I(n)$ get smaller and smaller. Knowing that Maple could not be trusted with floating point calculations (unless you change the value of `Digits` from its default, to something higher, say, in this case `Digits:=30`), that they get smaller and smaller. Typing '`evalf(L,30);`', Alex got:

$$[0.0651977994553206905827545006, 0.00374727011630229297588816663,$$
$$0.000247728866269394110526059, 0.0000176271312720269913734 7,$$
$$0.00000131246346593146768 53, 0.000000100776323486001254,$$
$$0.00000000791212964371946, 0.000000000631743771120 6,$$
$$5.1111100706 \times 10^{-11}, 4.17922459 \times 10^{-12}] \quad .$$

Alex realized that $I(n)$ seems to go to *zero* fairly fast, and since $I(10)/I(9)$ and $I(9)/I(8)$ were pretty close, Alex conjectured that the limit of $I(n)/I(n-1)$ tends to a certain constant. But ten data points do not suffice!

When Alex tried to find the first 2000 terms, Maple got slower and slower. Then Alex asked Alexa, the famous robot,

Alexa: how do I compute many terms of the sequence $I(n)$ given by that double-integral?

and Alexa replied:

Go to Doron Zeilberger's web-site and download the amazing program

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/MultiAlmkvistZeilberger.txt` , that accompanied the article [ApaZ]. Typing

```
MAZ(1,1/(1-x*y),x*(1-x)*y*(1-y)/(1-x*y),[x,y],n,N, {})[1];
```

immediately gave a recurrence satisfied by $I(n)$

$$I(n) = -\frac{\left(11\,n^2 - 11\,n + 3\right)}{n^2} \cdot I\left(n-1\right) + \frac{\left(n-1\right)^2}{n^2} \cdot I\left(n-2\right) \quad .$$

Using this recurrence, Alex easily computed the first 2000 terms, using the following Maple one-liner (calling the sequence defined by the recurrence `I2(n)`):

```
I2:=proc(n) option remember:  if n=0 then Pi**2/6 elif n=1 then 5-Pi**2/2 else -(11*n**2-
11*n+3)/n**2*I2(n-1)+(n-1)**2/n**2*I2(n-2):fi:  end:
```

and found out that indeed $I(n)/I(n-1)$ tends to a limit, about 0.09016994. Writing

$$I(n) = a_n - b_n \frac{\pi^2}{6}$$

and realizing that $I(n)$ is small, Alex found terrific **rational approximations** to $\frac{\pi^2}{6}$, a_n/b_n, that after **clearing denominators** can be written as a'_n/b'_n where *now* **both** numerator a'_n and denominator b'_n are **integers**.

$$\frac{\pi^2}{6} \approx \frac{a'_n}{b'_n} \quad .$$

Alex also noticed that for all n up to 2000, for some constant C,

$$|\frac{\pi^2}{6} - \frac{a'_n}{b'_n}| \leq \frac{C}{(b'_n)^{1+\delta}} \quad ,$$

where δ is roughly 0.09215925467. Then Alex concluded that this proves that $\frac{\pi^2}{6}$ is irrational, since if it were rational the left side would have been $\geq \frac{C_1}{b'_n}$, for some constant C_1. Of course, some details would still need to be filled-in, but that was not too hard.

The General Strategy

Let's follow Alex's lead. (Of course our fictional Alex owes a lot to the real Beukers and also to Alladi and Robinson [AR]).

Start with a constant, let's call it C, given by an explicit integral

$$\int_0^1 K(x)\,dx \quad ,$$

for some **integrand** $K(x)$, or, more generally, a d-dimensional integral

$$\int_0^1 \dots \int_0^1 K(x_1, \dots, x_k)\,dx_1 \dots dx_k \quad .$$

Our goal in life is to prove that C is irrational. Of course C may turn out to be rational (that happens!), or more likely, an algebraic number, or expressible in terms of a logarithm of an algebraic number, for which, there already exist irrationality proofs (albeit not always *effective* ones). But who knows? Maybe this constant has never been proved irrational, and if it will happen to be famous (e.g. Catalan's constant, or $\zeta(5)$, or the Euler-Mascheroni constant mentioned above), we will be famous too. But even if it is a nameless constant, it is still extremely interesting, if it is the *first* irrationality proof, since these proofs are so hard, witness that, in spite of great efforts by experts like Wadim Zudilin, the proofs of these are still wide open.

In this article we will present numerous candidates. Our proofs of irrationality are modulo a 'divisibility lemma' (see below), that we are *sure* that someone like Wadim Zudilin, to whom this paper is dedicated, can fill-in. Our only doubts are whether these constants are not already proved to be irrational because they happen to be algebraic (probably not, since Maple was unable to *identify* them), or more complicated numbers (like logarithms of algebraic numbers). Recall that Maple's `identify` can't (yet) identify *everything* that God can.

Following Beukers and Alladi-Robinson, we introduce a sequence of integrals, parameterized by a non-negative integer n

$$I(n) = \int_0^1 K(x)\,(x(1-x)K(x))^n\,dx \quad ,$$

and analogously for multiple integrals, or more generally

$$I(n) = \int_0^1 K(x)\,(x(1-x)S(x))^n\,dx \quad ,$$

for another function $S(x)$. Of course $I(0) = C$, our constant that we want to prove irrational.

It so happens that for a wide class of functions $K(x)$, $S(x)$, (for single or multivariable x) using the **Holonomic ansatz** [Ze1], and implemented (for the single-variable case) in [AlZ], and for the multi-variable case in [ApZ], and *much more efficiently* in [K], there exists a **linear recurrence equation with polynomial coefficients**, that can be actually computed (always in theory, but also often in practice, unless the dimension is high). In other words we can find a positive integer L, the **order** of the recurrence, and **polynomials** $p_0(n), p_1(n), \ldots, p_L(n)$, such that

$$p_0(n)I(n) + p_1(n)I(n+1) + \ldots + p_L(n)I(n+L) \;=\; 0 \quad .$$

If we are lucky (and all the cases in this paper fall into this case) the order L is 2. Furthermore, it would be evident in all the examples in this paper that $p_0(n), p_1(n), p_2(n)$ can be taken to have **integer coefficients**.

Another 'miracle' that happens in all the examples in this paper is that $I(0)$ and $I(1)$ are rationally-related, i.e. there exist integers c_0, c_1, c_2 such that

$$c_0 I(0) + c_1 I(1) = c_2 \quad ,$$

that our computers can easily find.

It then follows, by induction, that one can write

$$I(n) = b_n C - a_n \quad ,$$

for some sequences of **rational numbers** $\{a_n\}$ and $\{b_n\}$ that both satisfy the same recurrence as $I(n)$.

Either using trivial bounds on the integral, or using the so-called Poincaré lemma (see, e.g. [vdP], [ZeZu1],[ZeZu2]) it turns out that

$$a_n = \Omega(\alpha^n) \quad , \quad b_n = \Omega(\alpha^n) \quad ,$$

for some constant $\alpha > 1$, and

$$|I(n)| = \Omega(\frac{1}{\beta^n}) \quad ,$$

for some constant $\beta > 1$.

[Please note that we use Ω in a looser-than-usual sense, for us $x(n) = \Omega(\alpha^n)$ means that $\lim_{n\to\infty} \frac{\log x(n)}{n} = \alpha$.]

In the tweaks of Beukers' integrals for $\zeta(2)$ and $\zeta(3)$ coming up later, α and β are equal, but in the tweaks of the Alladi-Robinson integrals, α is usually different than β.

It follows that

$$|C - \frac{a_n}{b_n}| = \Omega(\frac{1}{(\alpha\beta)^n}) \quad .$$

Note that a_n, and b_n are, usually, **not** integers, but rather *rational numbers* (In the original Beukers/Apéry cases, the b_n were integers, but the a_n were not, in the more general cases in this article, usually neither of them are integers).

It so happens, in all the cases that we discovered, that there exists another sequence of rational numbers $E(n)$ such that

$$a'_n := a_n \, E(n) \quad , \quad b'_n := b_n \, E(n) \quad ,$$

are always **integers**, and, of course $gcd(a'_n \, , \, b'_n) = 1$. We call $E(n)$ the **integer-ating factor**.

In some cases we were able to conjecture $E(n)$ exactly, in terms of products of primes satisfying certain conditions (see below), but in other cases we can only conjecture that such an explicitly-describable sequence exists.

In either case there exists a real number, that sometimes can be described exactly, and other times only estimated, let's call it ν, such that

$$\lim_{n\to\infty} \frac{\log E(n)}{n} = \nu \quad ,$$

or, in our notation, $E(n) = \Omega(e^{n\nu})$.

Since we have

$$|C - \frac{a'_n}{b'_n}| = \Omega(\frac{1}{(\alpha\beta)^n}) \quad ,$$

where $b'_n = \Omega(e^{\nu\, n} \alpha^n)$. We need a **positive** δ such that

$$(e^{\nu\, n} \alpha^n)^{1+\delta} = (\alpha\beta)^n \quad .$$

Taking log (and dividing by n) we have

$$(\nu + \log\alpha)(1+\delta) = \log\alpha + \log\beta \quad ,$$

giving

$$\delta = \frac{\log\beta - \nu}{\log\alpha + \nu} \quad .$$

If we are *lucky*, and $\log\beta > \nu$, then we have $\delta > 0$, and an irrationality proof!, Yea! We also, at the same time, determined an *irrationality measure* (see [vdP])

$$1 + \frac{1}{\delta} \; = \; \frac{\log\alpha + \log\beta}{\log\beta - \nu} \quad .$$

If we are *unlucky*, and $\delta < 0$, it is still an exponentially fast way to compute our constant C to any desired accuracy.

Summarizing: For each specific constant defined by a definite integral, we need to exhibit

• A second-oder recurrence equation for the numerator and denominator sequence a_n and b_n that feature in $I(n) = b_n C - a_n$.

• The *initial conditions* a_0, a_1, b_0, b_1 enabling a very fast computation of many terms of a_n, b_n.

• The constants α and β

• Exhibit a conjectured **integer-ating factor** $E(n)$, or else conjecture that one exists, and find, or estimate (respectively), $\nu := \lim_{n\to\infty} \frac{\log E(n)}{n}$.

• Verify that $\beta > e^{\nu}$ and get (potentially) famous.

The Three Classical Cases

$\log \mathbf{2}$ ([AR])

$$C \; = \; \int_0^1 \frac{1}{1+x}\, dx \; = \; \log 2 \quad .$$

$$I(n) = \int_0^1 \frac{(x(1-x))^n}{(1+x)^{n+1}}\, dx \quad .$$

Recurrence:

$$(n+1)\,X\,(n) + (-6\,n-9)\,X\,(n+1) + (n+2)\,X\,(n+2) \;=0 \quad .$$

$$\alpha \;=\; \beta = 3+2\sqrt{2} \quad .$$

Initial conditions

$$a_0 = 0\,,\; a_1 = 2 \quad ; \quad b_0 = 1\,,\; b_1 = 3 \quad .$$

Integer-ating factor $E(n) = lcm(1\ldots n)$, $\nu = 1$.

$$\delta = \frac{\log\beta-\nu}{\log\alpha+\nu} = \frac{\log\beta-1}{\log\alpha+1} = \frac{\log(3+2\sqrt{2})-1}{\log(3+2\sqrt{2})+1} = 0.276082871862633587 \quad .$$

Implied irrationality measure: $1+1/\delta = 4.622100832454231334\ldots$.

$\zeta(\mathbf{2})$ ([B])

$$C = \int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy \;=\; \zeta(2) \quad .$$

$$I(n) = \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}}\,dx\,dy \quad .$$

Recurrence:

$$-\left(1+n\right)^2 X\,(n) + \left(11\,n^2+33\,n+25\right) X\,(n+1) + \left(2+n\right)^2 X\,(n+2) \;=0 \quad .$$

$$\alpha \;=\; \beta = \frac{11}{2} + \frac{5\sqrt{5}}{2} \quad .$$

Initial conditions

$$a_0 = 0\,,\; a_1 = -5 \quad ; \quad b_0 = 1\,,\; b_1 = -3 \quad .$$

Integer-ating factor $E(n) = lcm(1\ldots n)^2$, $\nu = 2$.

$$\delta = \frac{\log\beta-\nu}{\log\alpha+\nu} = \frac{\log\beta-2}{\log\alpha+2} = \frac{\log(11/2+5\sqrt{5}/2)-2}{\log(11/2+5\sqrt{5}/2)+2} = 0.09215925473323\ldots \quad .$$

Implied irrationality measure: $1+1/\delta = 11.85078219105234269595 28\ldots$.

$\zeta(\mathbf{3})$ ([B])

$$C = \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-z+xyz}\, dx\, dy\; dz = \zeta(3) \quad .$$

$$I(n) = \int_0^1 \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y)z(1-z))^n}{(1-z+xyz)^{n+1}}\, dx\, dy\, dz \quad .$$

Recurrence:

$$(1+n)^3\, X\,(n) - (2\,n+3)\left(17\,n^2+51\,n+39\right) X\,(n+1) + (n+2)^3\, X\,(n+2)\; = 0 \quad .$$

$$\alpha \;=\; \beta = 17 + 12\,\sqrt{2} \quad .$$

Initial conditions

$$a_0 = 0\,,\; a_1 = 12 \quad ; \quad b_0 = 1\,,\; b_1 = 5 \quad .$$

Integer-ating factor $E(n) = lcm(1\ldots n)^3$, $\nu = 3$.

$$\delta = \frac{\log\beta - \nu}{\log\alpha + \nu} = \frac{\log\beta - 3}{\log\alpha + 3} = \frac{\log(17+12\,\sqrt{2}) - 3}{\log(17+12\,\sqrt{2}) + 3} = 0.08052943118906168518 6\ldots \quad .$$

Implied irrationality measure: $1 + 1/\delta = 13.41782023335376578458\ldots$.

Accompanying Maple packages

This article is accompanied by three Maple packages, `GenBeukersLog.txt`, `GenBeukersZeta2.txt`, `GenBeukersZeta3.txt` all freely available from the *front* of this masterpiece

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/beukers.html` ,

where one can find ample sample input and output files, that readers are welcome to extend.

Zudilin's Tweak of the Beukers $\zeta(2)$ integral to get the Catalan constant

The *inspiration* for our *tweaks* came from Wadim Zudilin's brilliant discovery [Zu1] that the famous Catalan constant, that may be defined by the innocent-looking alternating series of the reciprocals of the odd perfect-squares

$$C := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \quad ,$$

can be written as the double integral

$$\frac{1}{8} \int_0^1 \int_0^1 \frac{x^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}}{1-xy} \, dx \, dy \quad .$$

This lead him to consider the sequence of Beukers-type double-integrals

$$I(n) \; = \int_0^1 \int_0^1 \frac{x^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}}{1-xy} \cdot \left(\frac{x(1-x)y(1-y)}{1-xy} \right)^n dx \, dy \quad .$$

Using the Zeilberger algorithm, Zudilin derived a three term recurrence for $I(n)$ leading to good diophantine approximations to the Catalan constant, alas not good enough to prove irrationality. This was elaborated and extended by Yu. V. Nesterenko [N]. See also [Zu2].

Using the multivariable Almkvist-Zeilberger algorithm we can derive the recurrence much faster. Using Koutschan's package [K], it is yet faster.

Our Tweaks

Inspired by Zudilin's Beukers-like integral for the Catalan constant, we decided to use our efficient tools for quickly manufacturing recurrences.

We systematically investigated the following families.

Generalizing the Alladi-Robinson-Like Integral for $\log 2$

Alladi and Robinson [AR] gave a Beukers-style new proof of the irrationality of $\log 2$ using the elementary fact that

$$\log 2 \; = \int_0^1 \frac{1}{1+x} \, dx \quad ,$$

and more generally,

$$\frac{1}{c} \log(1+c) \; = \int_0^1 \frac{1}{1+cx} \, dx \quad .$$

They used the sequence of integrals

$$I(n) := \int_0^1 \frac{1}{1+cx} \left(\frac{x(1-x)}{1+cx} \right)^n dx \quad ,$$

and proved that for a wide range of choices of rational c, this leads to irrationality proofs and irrationality measures (see also [ZeZu1]).

Our generalized version is the three-parameter family of constants

$$I_1(a,b,c) := \frac{1}{B(1+a,1+b)} \int_0^1 \frac{x^a(1-x)^b}{1+cx} \, dx$$

that is easily seen to equal ${}_2F_1(1, a+1; a+b+2; -c)$.

We use the sequence of integrals

$$I_1(a,b,c)(n) := \frac{1}{B(1+a,1+b)} \int_0^1 \frac{x^a(1-x)^b}{1+cx} \cdot \left(\frac{x(1-x)}{1+cx}\right)^n \, dx \quad .$$

Using the (original!) Almkvist-Zeilberger algorithm [AlZ], implemented in the Maple package

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt` ,

we immediately get a second-order recurrence that can be gotten by typing '`OpeL(a,b,c,n,N);`' in the Maple package

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/GenBeukersLog.txt` .

This enabled us to conduct a systematic search, and we found many cases of ${}_2F_1$ evaluations that lead to irrationality proofs, i.e. for which the δ mentioned above is **positive**. Many of them turned out to be (conjecturally) expressible in terms of algebraic numbers and/or logarithms of rational numbers, hence proving them irrational is not that exciting, but we have quite a few not-yet-identified (and inequivalent) cases. See the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenBeukersLog1.txt` ,

for many examples. Whenever Maple was able to (conjecturally) identify the constants explicitly, it is mentioned. If nothing is mentioned then these are potentially explicit constants, expressible as a hypergeometric series ${}_2F_1$, for which this would be the **first** irrationality proof, once the details are filled-in.

We also considered the four-parameter family of constants

$$I_1'(a,b,c,d) := \frac{\int_0^1 \frac{x^a(1-x)^b}{(1+cx)^{d+1}} \, dx}{\int_0^1 \frac{x^a(1-x)^b}{(1+cx)^{d}} \, dx} \quad ,$$

and, using the more general recurrence, also obtained using the Almkvist-Zeilberger algorithm (to see it type '`OpeLg(a,b,c,d,n,Sn);`' in `GenBeukersLog.txt`), found many candidates for irrationality proofs that Maple was unable to identify. See the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenBeukersLog2.txt` .

Generalizing the Beukers Integral for $\zeta(2)$

Define

$$I_2(a_1,a_2,b_1,b_2)(n) \; = \; \frac{1}{B(1-a_1,1-a_2)B(1-b_1,1-b_2)} \cdot$$

$$\int_0^1 \int_0^1 \frac{x^{-a_1}(1-x)^{-a_2} y^{-b_1}(1-y)^{-b_2}}{1-xy} \cdot \left(\frac{x(1-x)y(1-y)}{1-xy}\right)^n \, dx \, dy \quad ,$$

that happens to satisfy a linear-recurrence equation of second order, yielding Diophantine approximations to the constant $I_2(a_1, a_2, b_1, b_2)(0)$, let's call it $C_2(a_1, a_2, b_1, b_2)$

$$C_2(a_1, a_2, b_1, b_2) \,=\, \frac{1}{B(1-a_1, 1-a_2)B(1-b_1, 1-b_2)} \cdot \int_0^1 \int_0^1 \frac{x^{-a_1}(1-x)^{-a_2} y^{-b_1}(1-y)^{-b2}}{1-xy} \, dx\, dy \quad .$$

It is readily seen that

$$C_2(a_1, a_2, b_1, b_2) \,=\, {}_3F_2 \left(\begin{matrix} 1\,,\, 1-a_1\,,\, -b_1+1 \\ 2-a_1-a_2\,,\, 2-b_1-b_2 \end{matrix} ; 1 \right) \quad .$$

Most choices of random a_1, a_2, b_1, b_2 yield disappointing, negative δ's, just like $C_2(\frac{1}{2}, 0, 0, \frac{1}{2})$ (alias 8 times the Catalan constant), but a systematic search yielded several hundred candidates that produce positive δ's and hence would produce irrationality proofs. Alas, many of them were conjecturally equivalent to each other via a fractional-linear transformation with integer coefficients, $C \to \frac{a+bC}{c+dC}$, with a, b, c, d integers, hence the facts that they are irrational are equivalent. Nevertheless we found quite a few that are (conjecturally) *not equivalent* to each other. Modulo filling-in some details, they lead to irrationality proofs. Amongst them some were (conjecturally) identified by Maple to be either algebraic, or logarithms of rational numbers, for which irrationality proofs exist for thousands of years (in case of $\sqrt{2}$ and $\sqrt{3}$ etc.), or a few hundred years (in case of $\log 2$, etc.).

But some of them Maple was **unable** to identify, so *potentially* our (sketches) of proofs would be the **first** irrationality proofs.

Beukers $\zeta(2)$ Tweaks That produced Irrationality Proofs with Identified Constants

Denominator 2

We first searched for $C_2(a_1, a_2, b_1, b_2)$ where the parameters a_1, a_2, b_1, b_2 have denominator 2, there were quite a few of them, but they were all conjecturally equivalent to each other. Here is one of them:

- $C_2(0, 0, \frac{1}{2}, 0) = {}_3F_2(1, 1, 1/2; 2, 3/2; 1)$, alias $2 \log 2$.

Denominator 3

There were also quite a few where the parameters a_1, a_2, b_1, b_2 have denominator 3, but again they were all equivalent to each other, featuring $\pi\sqrt{3}$. Here is one of them.

- $C_2(0, 0, \frac{1}{3}, -\frac{2}{3}) = {}_3F_2(1, 1, 2/3; 2, 7/3; 1)$, alias (conjecturally) $-6 + 4\pi\sqrt{3}/3$.

Denominator 4

There were also quite a few where the parameters a_1, a_2, b_1, b_2 have denominator 4, but again they were all equivalent to each other, featuring $\sqrt{2}$, yielding a new proof of the irrationality of $\sqrt{2}$ (for what it is worth). Here is one of them.

• $C_2(-\frac{3}{4},-\frac{3}{4},-\frac{1}{4},-\frac{3}{4}) = {}_3F_2(1,7/4,5/4;7/2,3;1)$, alias (conjecturally) $-240\,+\,\frac{512}{3}\sqrt{2}$.

Denominator 5

There were also quite a few where the parameters a_1, a_2, b_1, b_2 have denominator 5, but again they were all equivalent to each other, featuring $\sqrt{5}$, yielding a new proof of the irrationality of $\sqrt{5}$ (for what it is worth). Here is one of them.

• $C_2(-\frac{4}{5},-\frac{4}{5},-\frac{2}{5},-\frac{3}{5}) = {}_3F_2(1,9/5,7/5;18/5,3;1)$, alias (conjecturally) $-\frac{845}{2}\,+\,\frac{2275}{12}\sqrt{5}$

Denominator 6 with identified constants

We found two equivalence classes where the parameters a_1, a_2, b_1, b_2 have denominator 6, for which the constants were identified. Here are one from each class.

• $C_2(-5/6,-5/6,-1/2,-1/2) = {}_3F_2(1,11/6,3/2;11/3,3;1)$, alias (conjecturally) $-\frac{1344}{5}+\frac{16384\sqrt{3}}{105}$

• $C_2(-5/6,-5/6,-1/3,-2/3) = {}_3F_2(1,11/6,4/3;11/3,3;1)$, alias (conjecturally) $\frac{972\,2^{2/3}}{5}-\frac{1536}{5}$

denominator 7 with identified constants

We found two cases where the parameters a_1, a_2, b_1, b_2 have denominator 7, for which the constants were identified.

• $C_2(-6/7,-6/7,-4/7,-3/7) = {}_3F_2(1,13/7,11/7;26/7,3;1)$, alias (conjecturally) the positive root of $13824\,x^3 - 2757888\,x^2 - 10737789048\,x + 16108505539 = 0$.

• $C_2(-6/7,-1/7,4/7,2/7) = {}_3F_2(1,13/7,3/7;3,8/7;1)$, alias (conjecturally) the positive root of $2299968\,x^3 + 7074144\,x^2 - 11234916\,x - 12663217 = 0$

Beukers $\zeta(2)$ Tweaks That produced Irrationality Proofs with Not-Yet-Identified Constants (and Hence Candidates for First Irrationality Proofs)

For the following constants, Maple was unable to identify, and we have *potentially* the first irrationality proofs of these constants.

Denominator 6 with not yet identified constants

We found two cases (up to equivalence):

• $C_2(0,-1/2,1/6,-1/2) = {}_3F_2(1,1,5/6;5/2,7/3;1)$

While Maple was unable to identify this constant, Mathematica came up with $-24\,-\,\frac{81\sqrt{\pi}\Gamma(7/3)}{\Gamma(-1/6)}$.

• $C_2(-2/3,-1/2,1/2,-1/2) = {}_3F_2(1,5/3,1/2;19/6,2;1)$

While Maple was unable to identify this constant, Mathematica came up with $\frac{13}{2}\,-\,\frac{6\Gamma(19/6)}{\sqrt{\pi}\Gamma(8/3)}$.

Denominator 7 with not yet identified constants

We found six cases (up to equivalence):

- $C_2(-6/7,-6/7,-4/7,-5/7) = {}_3F_2(1,13/7,11/7;26/7,23/7;1)$
- $C_2(-6/7,-5/7,-3/7,-5/7) = {}_3F_2(1,13/7,10/7;25/7,22/7;1)$
- $C_2(-6/7,-5/7,-2/7,-1/7) = {}_3F_2(1,13/7,9/7;25/7,17/7;1)$
- $C_2(-6/7,-4/7,-1/7,-1/7) = {}_3F_2(1,13/7,8/7;24/7,16/7;1)$
- $C_2(-6/7,-3/7,-5/7,-3/7) = {}_3F_2(1,13/7,12/7;23/7,22/7;1)$
- $C_2(-5/7,-3/7,-4/7,-2/7) = {}_3F_2(1,12/7,11/7;22/7,20/7;1)$

For each of them, to get the corresponding theorem and proof, use procedure `TheoremZ2` in the Maple pacgage `GenBeukersZeta2.txt`.

To get a statement and full proof (modulo a divisibility lemma) type , in `GenBeukersZeta2.txt`

```
TheoremZ2(a1,a2,b1,b2,K,0):
```

with `K` at least 2000. For example, for the last constant in the above list ${}_3F_2(1,12/7,11/7;22/7,20/7;1)$, type

```
TheoremZ2( -5/7, -3/7, -4/7, -2/7 ,3000,0):
```

For more details (the recurrences, the estimated irrationality measures, the initial conditions) see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenBeukersZeta2g.txt` .

Generalizing the Beukers Integral for $\zeta(3)$

The natural extension would be the **six**-parameter family (but now we make the exponents positive)

$$\frac{1}{B(1+a_1,1+a_2)B(1+b_1,1+b_2)B(1+c_1,1+c_2)} \cdot$$

$$\int_0^1\int_0^1\int_0^1 \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{1-z+xyz}\cdot\left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz}\right)^n dx\,dy\,dz \quad .$$

However, for **arbitrary** a_1,a_2,b_1,b_2,c_1,c_2 the recurrence is **third order**. (Wadim Zudilin pointed out that this may be related to the work of Rhin and Viola in [RV]).

Also, empirically, we did not find many promising cases. Instead, let's define

$$J_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(n)$$

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^e} \cdot \left(\frac{x(1-x)y(1-y)z(1-z)}{1-z+xyz}\right)^n \, dx\, dy\, dz \quad .$$

and

$$I_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(n) := \frac{J_3(a_1,a_2,b_1,b_2,c_1,c_2;e+1)(n)}{J_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(0)}$$

The family of constants that we hope to prove irrationality is the seven-parameter:

$$I_3(a_1,a_2,b_1,b_2,c_1,c_2;e)(0) \quad .$$

$$= \frac{\int_0^1 \int_0^1 \int_0^1 \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^{e+1}}\, dx\, dy\, dz}{\int_0^1 \int_0^1 \int_0^1 \frac{x^{a_1}(1-x)^{a_2}y^{b_1}(1-y)^{b_2}z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^{e}}\, dx\, dy\, dz} \quad .$$

Of course, for this more general, 7-parameter, family, there is no second-order recurrence, but rather a third-order one. But to our delight, we found a five-parameter family, let's call it

$$K(a,b,c,d,e)(n) := I_3(b,c,e,a,a,c,d)(n) \quad .$$

Spelled-out, our five-parameter family of constants is

$$K(a,b,c,d,e)(0) =$$

$$\frac{\int_0^1 \int_0^1 \int_0^1 \frac{x^{b}(1-x)^{c}y^{e}(1-y)^{a}z^{a}(1-z)^{c}}{(1-z+xyz)^{d+1}}\, dx\, dy\, dz}{\int_0^1 \int_0^1 \int_0^1 \frac{x^{b}(1-x)^{c}y^{e}(1-y)^{a}z^{a}(1-z)^{c}}{(1-z+xyz)^{d}}\, dx\, dy\, dz} \quad .$$

Now we found (see the section on finding recurrences below) a general second-order recurrence, that is too complicated to display here in full generality, but can be seen by typing

```
OPEZ3(a,b,c,d,e,n,Sn);
```

In the Maple package `GenBeukersZeta3.txt`. This enabled us, for each specific, numeric specialization of the parameters a, b, c, d, e to quickly find the relevant recurrence, and systematically search for those that give positive δ. Once again, many of them turned out to be (conjecturally) equivalent to each other.

Denominator 2:

We only found one class, up to equivalence, all related to $\log 2$. One of them is

$$K(0,0,0,1/2,1/2) = I_3(0,0,1/2,0,0,0,1/2) \quad ,$$

that is not that exciting since it is (conjecturally) equal to $-\frac{2-4\,\log(2)}{3-4\,\log(2)}$.

For details, type `TheoremZ3(0,0,0,1/2,1/2,3000,0);` in `GenBeukersZeta3.txt` .

Denominator 3:

We found three inequivalent classes, none of them Maple was able to identify.

$$K(0,0,0,1/3,2/3) = I_3(0,0,2/3,0,0,0,1/3) \quad ,$$

for details, type `TheoremZ3(0,0,0,1/3,2/3,3000,0);` in `GenBeukersZeta3.txt`.

$$K(0,0,0,2/3,1/3) = I_3(0,0,1/3,0,0,0,2/3) \quad ,$$

for details, type `TheoremZ3(0,0,0,2/3,1/3,3000,0);` in `GenBeukersZeta3.txt`.

$$K(0,1/3,2/3,1/3,2/3) = I_3(0,0,1/3,0,0,0,2/3) \quad ,$$

for details, type `TheoremZ3(0,1/3,2/3,1/3,2/3,3000,0);` in `GenBeukersZeta3.txt`,

These three constants are **candidates** for **'first-ever-irrationality proof'**.

Denominator 4: We only found one family, all expressible in terms of $\log 2$. Here is one of them.

For example

$$K(0,1/2,0,1/4,3/4) = I_3(1/2,0,3/4,0,0,0,1/4) \quad ,$$

that, conjecturally equals $-\frac{-30+45\ \log(2)}{-11+15\ \log(2)}$.

For details, type `TheoremZ3(0,1/2,0,1/4,3/4,3000,0);` in `GenBeukersZeta3.txt`.

Denominator 5: We only found one family, up to equivalence, but Maple was unable to identify the constant. So it is potentially the first irrationality proof of that constant

$$K(0,1/5,0,3/5,2/5) = I_3(1/5,0,2/5,0,0,0,3/5) \quad .$$

For details, type `TheoremZ3(0,1/5,0,3/5,2/5,3000,0);` in `GenBeukersZeta3.txt`.

Denominator 6: We found three families, up to equivalence, none of which Maple was able to identify. Once again, these are candidates for **first-ever irrationality proofs** for these constants.

$$K(0,1/2,1/2,1/3,1/6) = I_3(1/2,1/2,1/6,0,0,1/2,1/3) \quad .$$

For details, type `TheoremZ3(0,1/2,1/2,1/3,1/6,3000,0);` in `GenBeukersZeta3.txt`.

$$K(0,1/2,1/2,1/6,1/3) = I_3(1/2,1/2,1/3,0,0,1/2,1/6) \quad .$$

For details, type `TheoremZ3(0,1/2,1/2,1/6,1/3,3000,0);` in `GenBeukersZeta3.txt`.

$$K(1/3,0,2/3,1/2,5/6) = I_3(0,2/3,5/6,1/3,1/3,2/3,1/2) \quad .$$

For details, type `TheoremZ3(1/3,0,2/3,1/2,5/6,3000,0);` in `GenBeukersZeta3.txt`.

Denominator 7: We found five families, up to equivalence, none of which Maple was able to identify. Once again, these are candidates for first-ever irrationality proofs for these constants.

$$K(1/7,0,2/7,3/7,4/7) = I_3(0,2/7,4/7,1/7,1/7,2/7,3/7) \quad .$$

For details, type `TheoremZ3(1/7,0,2/7,3/7,4/7,3000,0);` in `GenBeukersZeta3.txt`.

$$K(1/7,0,2/7,5/7,3/7) = I_3(0,2/7,3/7,1/7,1/7,2/7,5/7) \quad .$$

For details, type `TheoremZ3(1/7,0,2/7,5/7,3/7,3000,0);` in `GenBeukersZeta3.txt`.

$$K(1/7,0,3/7,4/7,5/7) = I_3(0,3/7,5/7,1/7,1/7,3/7,4/7) \quad .$$

For details, type `TheoremZ3(1/7,0,3/7,4/7,5/7,3000,0);` in `GenBeukersZeta3.txt`.

$$K(1/7,0,4/7,2/7,5/7) = I_3(0,4/7,5/7,1/7,1/7,4/7,2/7) \quad .$$

For details, type `TheoremZ3(1/7,0,4/7,2/7,5/7,3000,0);` in `GenBeukersZeta3.txt`.

$$K(2/7,0,3/7,4/7,5/7) = I_3(0,3/7,5/7,2/7,2/7,3/7,4/7) \quad .$$

For details, type `TheoremZ3(2/7,0,3/7,4/7,5/7,3000,0);` in `GenBeukersZeta3.txt`.

If you don't have Maple, you can look at the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenBeukersZeta3All.txt` ,

that gives detailed sketches of irrationality proofs of all the above constants, some with conjectured integer-ating factors.

Guessing an INTEGER-ating factor

In the original Beukers cases the integer-ating factor was easy to conjecture, and even to prove. For $\zeta(2)$ it was $lcm(1\dots n)^2$, and for $\zeta(3)$ it was $lcm(1\dots n)^3$. For the Alladi-Robinson case of $\log 2$ it was even simpler, $lcm(1\dots n)$.

But in other cases it is much more complicated. A natural 'atomic' object is, given a modulo M, a subset C of $\{0,...,M-1\}$, rational numbers e_1, e_2 between 0 and 1, rational numbers e_3, e_4, the following quantity, for positive integers n

$$Pp(e_1, e_2, e_3, e_4, C, M; n) := \prod_p p \quad ,$$

where p ranges over all primes such that (let $\{a\}$ be the fractional part of a, i.e. $a - \lfloor a \rfloor$)

- $e_1 < \{n/p\} < e_2$
- $e_3 < p/n < e_4$
- $p \; mod \; M \in C$

Using the prime number theorem, it follows (see e.g. [Zu2]) that

$$\lim_{n \to \infty} \frac{\log Pp(e_1, e_2, e_3, e_4, C, M; n)}{n} \quad ,$$

can be evaluated exactly, in terms of the function $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ (see procedure `PpGlimit` in the Maple packages) thereby giving an exact value for the quantity δ whose positivity implies irrationality.

Of course, one still needs to rigorously prove that the conjectured integer-ating factor is indeed correct.

Looking under the hood: On Recurrence Equations

For 'secrets from the kitchen' on how we found the second-order, four-parameter recurrence operator
`OPEZ2(a1,a2,b1,b2,n,N)` in the Maple package `GenBeukersZeta2.txt`, that was the engine driving the $\zeta(2)$ tweaks, and more impressively, the five-parameter second-order recurrence operator `OPEZ3(a,b,c,d,e,n,N)` in the Maple package `GenBeukersZeta3.txt`, that was the engine driving the $\zeta(3)$ tweaks, the reader is referred to the stand-alone appendix available from the following url:

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/beukersAppendix.pdf` .

Other Variations on Apéry's theme

Other attempts to use Apéry's brilliant insight are [Ze2][Ze3][ZeZu1]. Recently Marc Chamberland and Armin Straub [CS] explored other fascinating aspects of the Apéry numbers, not related to irrationality.

Conclusion and Future Work

We believe that symbolic computational methods have great potential in irrationality proofs, in particular, and number theory in general. In this article we confined attention to approximating sequences that arise from *second-order recurrences*. The problem with higher order recurrences is that one gets linear combinations with rational coefficients of *several* constants, but if you can get two different such sequences coming from third-order recurrences, both featuring the same two constants, then the present method may be applicable. More generally if you have a k-th order recurrences, you need $k-1$ different integrals.

The general methodology of this article can be called *Combinatorial Number Theory*, but not in the usual sense, but rather as an analog of *Combinatorial Chemistry*, where one tries out many potential chemical compounds, most of them useless, but since computers are so fast, we can afford to generate lots of cases and pick the wheat from the chaff.

Encore: Hypergeometric challenges

As a *tangent*, we (or rather Maple) discovered many exact ${}_3F_2(1)$ evaluations. Recall that the Zeilberger algorithm can prove hypergemoetric identities only if there is at least one free parameter. For a *specific* ${}_3F_2(a_1\, a_2\, a_3\,; b_1\, b_2; 1)$, with *numeric* parameters, it is useless. Of course, it is sometimes possible to introduce such a parameter in order to conjecture a *general* identity, valid for 'infinitely' many n, and then specialize n to a specific value, but this remains an art rather than a science. The output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenBeukersZeta2f.txt`

contains many such conjectured evaluations, (very possibly many of them are equivalent via a hypergeometric transformation rule) and we challenge Wadim Zudilin, the *birthday boy*, or anyone else, to prove them.

Happy Ending

The *birthday boy* brilliantly met the challenges! See his brilliant note [Zu3].

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