An Irrationality measure of arctan(1/p)

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Abstract. We establish an irrationality measur for $\arctan(\frac{1}{p})$ for any odd prime p.

The Maple package. This article is accompanied by a shoft Maple package

- `ArcTan.txt` ;

It is available, along with an input and output file, from the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/arctan.html` .

Let p be an odd prime. Consider the following integral

$$E_n(p) \, = \, \int_0^1 \frac{x^n\,(1-x)^n}{(1+\frac{x^2}{p^2})^{n+1}}\, dx \quad .$$

Fact 1: There exists a constant C_1 such that

$$|E_n(p)| \, \leq \, \frac{C}{a_1(p)^n} \quad ,$$

where

$$a_1(p) \, = \, 2\, \frac{1}{\left(-p+\sqrt{1+p^2}\right)p} \quad .$$

Proof: Differentiating $f(x) \, = \, \frac{x(1-x)}{1+\frac{x^2}{p^2}}$ with respect to x and setting it equal to 0 gets that the maximum is at $x = \left(-p+\sqrt{1+p^2}\right)p$. Plugging it into $f(x)$ gives $1/a_1(p)$.

Fact 2: $E_n(p)$ satisfies the linear recurrence equation

$$(4n+8)E_{n+2}(p) - 2p^2(2n+3)E_{n+1}(p) - (n+1)p^2E_n(p) = 0 \quad .$$

Proof: Follows from the Almkvist-Zeilberger algorithm [AZ].

Fact 3: There exist rational numbers $A_n(p)$ and $B_n(p)$ such that

$$E_n(f) \, = \, -A_n \, + \, B_n\, arctan(\frac{1}{p}) \quad .$$

Proof: Either by integration by parts, or using Fact 2, once it is verified for $n=0$ and $n=1$.

Fact 4: The sequence of rational numbers $A_n(p)$, $B_n(p)$ satisfy the same recurrence as of $E_n(p)$, namely

$$(4n+8)A_{n+2}(p) + 2p^2(2n+3)A_{n+1}(p) - (n+1)p^2 A_n(p) = 0 \quad ,$$
$$(4n+8)B_{n+2}(p) + 2p^2(2n+3)B_{n+1}(p) - (n+1)p^2 B_n(p) = 0 \quad .$$

Proof: Use fact 3, comparing the coefficients of 1 and $\arctan(\frac{1}{p})$.

Fact 5: There exist constants C_2, C_3 such that

$$A_n(p) \leq C_2 a_2(p)^n \quad , \quad B_n(p) \leq C_3 a_2(p)^n \quad ,$$

where

$$a_2(p) \; = \; \frac{1}{2}\left(p + \sqrt{1+p^2}\right) p$$

Proof: From Fact 4 and the Poincaré Lemma.

Fact 6: There exists a constant C_4 such that

$$|arctan(\frac{1}{p}) \; - \; \frac{A_n(p)}{B_n(p)}| \leq \frac{C_4}{(a_1(p)\, a_2(p))^n} \quad .$$

Proof: Divide Fact 3 by B_n and use Facts 1 and 5.

Surprising Fact 7: Let $d(n) := lcm(1...n)$. Then

$$\bar{A}_n(p) = A_n(p)\, d(n)\, 2^{[3n/2]}/p^n \quad , \quad \bar{B}_n(p) = B_n(p)\, d(n)\, 2^{[3n/2]}/p^n$$

are integers.

Proof: Checked empirically for many p and many n. Formal proof coming up.

Fact 8:

$$\bar{B}_n(p) \; = \; O((2^{3/2}\, a_2(p)/p)^n d(n)) \quad ,$$

Proof: From Fact 5 and the definition of $\bar{B}_n(p)$.

From Fact 6 we have

Fact 9

$$|arctan(\frac{1}{p}) \; - \; \frac{\bar{A}_n}{\bar{B}_n}| \leq \frac{C_4}{(a_1(p)\, a_2(p))^n} = \frac{C_5}{(\bar{B}_n(p))^{1+\delta}} \quad , \quad ,$$

where (since $d(n) = O(e^n)$),

$$1 + \delta = \frac{\log(a_1(p)) + \log(a_2(p))}{\frac{3}{2}\log 2 + \log(a_2(p)) - \log(p) + 1} \quad .$$

As usual (see [vdP]), the irrationality measure is $1 + 1/\delta$, hence, we have established

Theorem: If p is an odd prime, then $arctan(\frac{1}{p})$ has irrationality measure

$$-2 \left(\ln \left(2\,\frac{p+\sqrt{1+p^2}}{p} \right) + \ln \left(1/2\, \left(p+\sqrt{1+p^2} \right) p \right) \right) \left(-2\, \ln \left(2\,\frac{p+\sqrt{1+p^2}}{p} \right) + 3\, \ln(2) + 2 - 2\, \ln(p) \right)^{-1}$$

Here are a few sample values

$p = 3 : 7.707357546$, $p = 5 : 4.788327432$, $p = 7 : 4.075567842$, $p = 11 : 3.542000040$,

$p = 1009 : 2.429989300$, $p = 1000000007 : 2.134188776$, $p = 100000000000000000039 : 2.059322676$.

References

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