The number of 1...d-avoiding permutations of length d+r for SYMBOLIC d but numeric r

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Dedicated to Ira Martin Gessel (b. April 9, 1951), on his millionth$_2$ birthday

Preface: How many permutations are there of length googol+30 avoiding an increasing subsequence of length googol?

This number is way too big for our physical universe, but the number of permutations of length googol+30 that *contain* at least one increasing subsequence of length googol is

3769987628815905643852921525646105664146833823621994801456991357113502936781270538054719048039675278
0769193354371721350001524610578097700045972792823897290959624203896101981952929640805170129282073883
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0989825934469696079344723053679001130033667827524966034661782064851068214182454731365743413486729730
0631055444127725930013792836515384850702346797298406803049230145697433567004811555984158378611125895
0145768901348725550726037527669812626353266837685037397408862767082018239579392663024131792105407280
4788720840618514463465035392103884394981202007834724110094447116613439128758285044269471808502083275
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0695626913029699922648102091615529949414940648588048836485372758775808743231365617459515329190972398
7074543946415578728439994306071279654045146432379557541358408978156863172980419720839292761025261752
6805876626590163265795592248178664681630980893821587688413815206609216082514983787883386977226071420
21649147728993592578961422177700294482596740993986519357246959930668146505085270714475561150113747221
20887870047753358177316206263356927955729458754686550644432634687680282027976402772772483836117105473
48145611509228154510472000404130614639780926417137329939732465722014680564902839930824306834920414545
13874753600055252092001136814571329384587325582468487887244395295245585419188646792764252832159962029
69411649544372131053235387439446875434698793735121412796400236965732584484687219982898355145980291977
86269234486135973112564250247007239135280355775712267954726019033893771378762777423669196575295174512
96452587669725726144832740371782822308006170531910099265678141483622517144014107716217010098383839968
84507804590244720667406593929564134591547805793634468523934454328906756758701209765471514880572370759
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08120140577436690007140701570599484176861047461052826774744899246746666909264578067076243923453088561
96698778069217767194382941365732112039412879713531991598317675682505439845424625600438225076973116586
49130213308514799728830764637172129004065611907475610401713008790972891409020362658741946509891832165
77016676670060012096109989093803820108650038852207775655317011335432185883307209708526943588264818977

3775738149186073685934586558285596632901636818878886042883326839132327059391308990152857750191809745634879121424762765606213101234688450096506147759256582735622079237519547939943470930166182921645804027125427981433886411676141783011905981747938788069443016253221099379127552822077917790224660044792584082446294959276134988131654342203869918382647352510707580950827477809341316822096398440902856636229390040215415882419358649518674355414801095047413826040824566345129789426039221842088797052981439573736696502233079308864944908955066124222663770097587204880225587795134251004323489264304276651259449876930894224575112270628402898275433738688545939162654357055516205161266436378837328045722669166090867956953927163081562519904030045933274931742332018704568957075002591805894557106029373427199758644919233869688590384289777698000212965155221948358771775977404379888129917495848357217867552935026201493389870312223251822518408158990271446362436501824274759908263581759373772458033768880934255069534236693503642535491880914435376674876432270204764414065561382212425100253695336680109353578878041405262772638139124792832164064839419602862651995996632545125266426235388963188384177665364612927059366114930625908539780241862926623393421168173669371424135263438461510848532070094781148761874414915822566817516932438525928455634363409372944818437842421507459176260334046758894630063276039591166623100092626550628336007090706434133266477977993771226318438820360547721116201480593775052297853562022592500472291873865767469994494740534790765914361805057941708749765216546018547704334563663220497822600180042427352634146022024254868372879917906503008302949451445090553172508996790329329093550087454853933917873519408569488210748631879883374585250820777287677645800280443076699166062637606763797777023540421219334461005282376299007226578307082023454514148089887463748610689381677459821466400715603888673197538425720238 .

Hence the number of permutations of length googol+30 *avoiding* an increasing subsequence of length googol is $(googol + 30)!$ *minus* the above small number.

Counting the "Bad Guys"

Recall that thanks to Robinson-Schenstead ([R][Sc]), the number of permutations of length n that do **not** contain an increasing sequence of length d is given by

$$G_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) < d}} f_\lambda^2 \quad ,$$

where λ denotes a typical *Young diagram*.

Hence the number of permutations of length n that **do** contain an increasing sequence of length d is

$$B_d(n) := \sum_{\substack{\lambda \vdash n \\ \#rows(\lambda) \geq d}} f_\lambda^2 \quad .$$

Since the total number of permutations of length n is $n!$ ([B]), if we know how to find $B_d(n)$, we would know immediately $G_d(n) = n! - B_d(n)$, at least if we leave $n!$ alone as a factorial, rather than spell it out.

Recall that the *Hook Length formula* (see [Wiki]) tells you that if λ is a Young diagram then

$$f_\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)} \quad ,$$

where the product is over all the n cells c of the Young diagram and $h(c)$ is its *hook length*. If $c=(i,j)$, then $h(c)=(\lambda_i - i) + (\lambda'_j - j) + 1$, where λ' is the *conjuagate* diagram, where the rows become columns and vice-versa.

Let r be a fixed integer, then for *symbolic* d, valid for $d \geq r-1$, any Young diagram with $d+r$ cells can be written, for some Young diagram μ with $\leq r$ cells as $\mu = (\mu_1, \dots, \mu_r)$ (where we add zeros to the end if the number of parts of μ is less than r)

$$\lambda = (1+\mu_1, \dots, 1+\mu_r, 1^{d-r+r'}) \quad ,$$

where $r' = r - |\mu|$. For such a λ, with at least d rows,

$$\prod_{c \in \lambda} h(c) = \left(\prod_{c \in \mu} h(c) \right) \cdot ((d+r'+\mu_1)(d+r'-1+\mu_2) \cdots (d+r'-r+1+\mu_r)) \cdot (d-r+r')! \quad .$$

Hence f_λ, that is $(d+r)!$ divided by the above, is a certain specific number times a certain polynomial in d. Since, for s specific, *numeric*, r, there are only *finitely* many Young diagrams with at most r cells, the computer can find all of them, compute the polynomial corresponding to each of them, square it, and add-up all these terms, getting an *explicit* **polynomial** expression, in the variable d, for $B_d(d+r)$, the number of permutations of length $d+r$ that *contain* an increasing subsequence of length d. As we said above, from this we can find $G_d(d+r) = (d+r)! - B_d(d+r)$, valid for *symbolic* d larger than $r-1$.

$\mathbf{B_d(d+r)}$ for r from 1 to 30

$$B_d(d+1) = d^2 + 1 \quad ,$$

$$B_d(d+2) = \frac{1}{2}\, d^4 + d^3 + \frac{1}{2}\, d^2 + d + 3 \quad ,$$

$$B_d(d+3) = \frac{1}{6}\, d^6 + d^5 + \frac{5}{3}\, d^4 + \frac{2}{3}\, d^3 + \frac{19}{6}\, d^2 + \frac{31}{3}\, d + 11 \quad ,$$

$$B_d(d+4) = 47 + \frac{1}{24}\, d^8 + \frac{1}{2}\, d^7 + \frac{395}{6}\, d + \frac{247}{6}\, d^2 + \frac{29}{24}\, d^4 + 9\, d^3 + \frac{25}{12}\, d^6 + \frac{19}{6}\, d^5 \quad ,$$

$$B_d(d+5) = 239 + \frac{1}{120}\, d^{10} + \frac{1}{6}\, d^9 + \frac{31}{24}\, d^8 + \frac{14}{3}\, d^7 + \frac{3981}{10}\, d + \frac{10459}{30}\, d^2 + \frac{653}{24}\, d^4 + \frac{959}{6}\, d^3 + \frac{823}{120}\, d^6 + \frac{67}{30}\, d^5 \quad .$$

For $B_d(d+r)$ for r from 6 up to 30, see
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64a` .

Sequences

The sequence $G_3(n)$ is the greatest *celeb* in the kingdom of combinatorial sequences, the super-famous **A000108** in Neil Sloane's legendary database ([Sl]). $G_4(n)$ is also moderately famous, **A005802**. $G_5(n)$ is **A047889**, $G_6(n)$ is **A047890**, $G_7(n)$ is **A052399** , $G_8(n)$ is **A072131**, $G_9(n)$ is **A072132**, $G_{10}(n)$ is **A072133**, $G_{11}(n)$ is **A072167**, But $G_d(n)$ for $d \geq 12$ are no longer there (for a good reason, enough is enough!). Using the polynomials $B_d(d+r)$, we computed the first

$2d+1$ terms of $G_d(n)$ for $d \leq 30$. See
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64b` .

But, this method can only go up to $2d+1$ terms of the sequence $G_d(n)$, and of course, the first d terms are trivial, namely $n!$. Can we find the first 100 terms (or whatever) for the sequences $G_d(n)$ for d up to 20, and beyond, **efficiently?**

Encore: Efficient Computer-Algebra Implementation of Ira Gessel's AMAZING Determinant Formula

Recall Ira Gessel's [G] famous expression for the generating function of $G_d(n)$, *canonized* in the *bible* ([W], p. 996, Eq. (5)). Here it is:

$$\sum_{n \geq 0} \frac{G_d(n)}{n!^2}\, x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\dots,d} \quad ,$$

in which $I_\nu(t)$ is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\,t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Can we use this to compute the first 100 terms, of say, $G_20(n)$?

While computing *numerical* determinants is very fast, computing *symbolic* ones is a different story. First, do not get scared by the "infinite" power series. If we are only interested in the first N terms of $G_d(n)$, then it is safe to truncate the series up to t^{2N}, and take the determinant of a $d \times d$ determinants with *polynomial entries*. If you use the *vanilla* determinant in a computer-algebra system such as Maple, it would be very inefficient, since the degree of the determinant is much larger than $2N$. But a little of cleverness can make things more efficient. The Maple package `Gessel64`, available free of charge from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Gessel64`

accompanying this article, has a procdure `SeqIra(k,N)` that computes the first N terms of $G_k(n)$, using the following clever implementation of Gessel's famous formula

```
SeqIra:=proc(k,N) local gu,t,i,j, R:

R := table():

R[`0`] := 0:

R[`1`] := 1:

R[`+`] := `+`:

R[`-`] := `-`:
```

```
R[‘*‘] := proc(p, q):  return add(coeff(p*q, t, i)*t**i, i=0..2*N): end:

R[‘=‘] := proc(p, q):  return evalb(p = q):  end:

gu:=expand(LinearAlgebra[Generic][Determinant][R](Matrix([seq([seq(Iv(abs(i-j),t,2*N),j=1..k-
1)],i=1..k-1)]))):

[seq(coeff(gu,t,2*i)*i!**2,i=1..N)]:

end:
```

In the above code, procedure `Iv(v,t,N)` computes the truncated modified Bessel function that shows up in Gessel's determinant, and it is short enough to reproduce here.

```
Iv:=proc(v,t,N) local j:

add(t**(2*j+v)/j!/(j+v)!,j=0..trunc((N-v)/2)+1):

end:
```

Using this procedure, the first-named author computed (in 4507 seconds) the first 100 terms of each of the sequences $G_d(n)$ for $3 \leq d \leq 20$, and could have gone much further.

See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64c` .

HAPPY 64th BIRTHDAY, IRA!

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