The $O(1/n^{85})$ Asymptotic expansion of OEIS sequence A85

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In honor of our two heroes, Don Knuth who was 85*, a year ago, and Neil Sloane, who is now* 85

On Oct. 10, 2024, **exactly** 85 years after his birth, guru Neil Sloane gave a great Zoom talk: `https://vimeo.com/1018688642?share=copy` .

After the talk was over his many fans discussed the significance of the number 85, and someone mentioned that the OEIS sequence A85 (`https://oeis.org/A000085`) is a very important sequence, the number of involutions of length n, and also, thanks to the famous RSK (Robinson-Schenstead-KNUTH) correspondence, the number of Young tableaux with n boxes. And indeed this sequence was important enough that guru Don Knuth, in his *bible*, The Art of Computer Programming ([Kn], pp. 62-64) spent three pages deriving, *by hand*, using *Laplace's method*, the asymptotic formula for the members of $A85$, first determined, using a different method by Moser and Wyman ([MW]). Namely, $t_n := A85[n]$ (Eq. (53), p. 64, in [Kn]) is given by the following formula.

$$t_n \,=\, \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4} \left(1+\frac{7}{24}n^{-1/2}+O(n^{-3/4})\right) \quad .$$

Knuth remarked:

In principle*, the method we have used could be extended to any* $O(n^{-k})$*, for* **any** k*, instead of* $O(n^{-3/4})$*.*

In this modest tribute to our two heroes, we will do it **in practice** for the important number $k = 85$. Ideally one should be able to use the Maple package `http://sites.math.rutgers.edu/~zeilberg/tokhniot/AsyRec.txt` written by the third author, and executed by the first author, that, alas, for some mysterious reason, can only go as far as $k = 22$. Its extension, written in Sage by the second author, using the method in [KaJJ], and executed by his computer, can go as far as one wishes, but in honor of the two gurus we will stop at $O(1/n^{85})$.

We have (to order $O(n^{-5})$)

Theorem:

$$t_n \,=\, \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4} \cdot$$

$$(1+\frac{7}{24\sqrt{n}}-\frac{119}{1152n}-\frac{7933}{414720n^{\frac{3}{2}}}+\frac{1967381}{39813120n^2}-\frac{57200419}{1337720832n^{\frac{5}{2}}}-\frac{562799}{47775744n^3}-\frac{526420847}{40131624960n^{\frac{7}{2}}}$$
$$+\frac{1856209}{573308928n^4}-\frac{267645803}{2407897497600n^{\frac{9}{2}}}+O(\frac{1}{n^5})) \quad .$$

To get the $O(\frac{1}{n^{85}})$ asymptotics, that is too long to be typyset in *humanese*, see the output file:

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/A85asympt.txt` .

Comments

1. Our method is different from both the one used by Moser and Wyman and the one used by Knuth (that he attributes to Laplace). We use the method exposited in [WZ], originally invented by George D. Birkhoff and W. J. Trjitzinsky, and implemented by us in Maple and Sage. We use the fact, mentioned in [Kn] (p. 62, Eq. (40) there), that t_n (i.e. $A85[n]$) satisfies the second-order recurrence

$$t_n = t_{n-1} + (n-1)t_{n-2} \quad .$$

2. The combinatorial proof given in [Kn] (p. 62), goes like this:

A permutation is its own inverse if and only if its cycle form consists of one-cycles (fixed points) and two cycles (transpositions). Since t_{n-1} of the t_n involutions have (n) as a one-cycle, and since t_{n-2} of them have (jn) as a two-cyle, for fixed $j < n$, we obtain this formula.

We have two additional proofs, for what there are worth. Eq, (41) of ([Kn], p.62) says:

$$t_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!} \quad ,$$

(as explained in [Kn], suppose there are k two-cycles and $n-2k$ one-cycles. There are $\binom{n}{2k}$ ways to choose the fixed points, and the multinomial coefficient $(2k)!/(2!)^k$ is the number of ways to arrange the other elements into k distinguished transpositions; dividing by $k!$ to make the transpositions indistinguishable.)

Now download the Maple package:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt` ,

and type

`zeil(n!/((n-2*k)!*2**k*k!),k,n,N);` ,

and you would get the above recurrence, followed by its *proof certificate.*

Eq. (42) in [Kn], p. 62,

$$\sum_n t_n z^n/n! = e^{z+z^2/2} \quad ,$$

yields yet-another proof. In the above-mentioned Maple package, type

`AZd(n!*exp(z+z**2/2)/z**(n+1),z,n,N);` , and you would get the recurrence, followed by its *proof certificate*.

3. This article is accompanied by a Maple package `A85.txt` available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/A85.txt` ,

that lets you find the asymptotic expansion to order $O(1/n^{(k+1)/2})$ for *any* $1 \leq k \leq 169$. Just type

'`AsyI(n,k);`' if n is numeric, and '`AsyIs(n,k);`', if n is symbolic.

Procedures `SeqWn(N)` outputs the list of first N terms of OEIS sequence `A85`.

Just for fun, to get $A85[1000]$, (the number of involutions of length 1000) type:

`SeqWn(1000)[1000];` ,

getting, *right away*, the number in the following output file:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/A85w1000.txt` .

This is a 1296-digit number, that in floating point to twenty digits is:

$2.1439289538422655419 \cdot 10^{1296}$.

The approximation from using $O(1/n)$ asymptotics is gotten by typing

`evalf(AsyI(1000,1),20);`

yielding

$$2.1441496003431008422 \cdot 10^{1296} \quad .$$

The ratio to the exact value is:

$$1.0001029168902448312$$

On the other hand with `evalf(AsyI(1000,30),20);` (the $O(1/n^{31/2})$) formula), the ratio is:

$$0.99999999999999999999999999996 \ldots \quad .$$

Pas mal! .

References

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[WZ] Jet Wimp and Doron Zeilberger, *Resurrecting the Asymptotics of Linear Recurrences*, J. of Math. Anal. Appl. **111** (1985), 162-176.

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