Theorem 1: For any prime p and positive integer r, define

$$A(r;p) := \left(\sum_{n=0}^{rp-1} \binom{2n}{n}\right) \bmod p \ .$$

Then, $A(1;p) = \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3 \ . \end{cases}$

Proof: Observe that

$$\binom{2n}{n} = CT_x\left[\frac{(1+x)^{2n}}{x^n}\right]$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{p-1} \binom{2n}{n} &= \sum_{n=0}^{p-1} CT_x\left[\left(\frac{(1+x)^{2n}}{x^n}\right)\right] \\
&= \sum_{n=0}^{p-1} CT_x\left[\left(2+x+\frac{1}{x}\right)^n\right] \\
&= CT_x\left[\frac{\left(2+x+\frac{1}{x}\right)^p - 1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT_x\left[\frac{1+x^p+\frac{1}{x^p}}{1+x+\frac{1}{x}}\right] \\
&\equiv_p CT_x\left[\frac{1+x^p+x^{2p}}{(1+x+x^2)x^{p-1}}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1}{1+x+x^2}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1-x}{1-x^3}\right]
\end{aligned}$$

The result folllows from series expansion of the last expression.

Corollary 1: $A(2;p) = \begin{cases} 3, & \text{if } p \equiv 1 \mod 3 \\ -3, & \text{if } p \equiv 2 \mod 3 \end{cases}.$

Proof:

$$\begin{aligned}
\sum_{n=0}^{2p-1}\binom{2n}{n} &= \sum_{n=0}^{2p-1} CT_x\left[\left(\frac{(1+x)^{2n}}{x^n}\right)\right] \\
&= CT_x\left[\frac{\left(2+x+\frac{1}{x}\right)^{2p}-1}{2+x+\frac{1}{x}-1}\right] \\
&= CT_x\left[\frac{\left(6+4x+\frac{4}{x}+x^2+\frac{1}{x^2}\right)^{p}-1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT_x\left[\frac{\left(6+4x^p+\frac{4}{x^p}+x^{2p}+\frac{1}{x^{2p}}\right)-1}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{x^{2p-1}}\left[\frac{1+4x^p}{1+x+x^2}\right] \\
&\equiv_p COEFF_{x^{2p-1}}\left[\frac{1}{1+x+x^2}\right] + 4COEFF_{x^{p-1}}\left[\frac{1}{1+x+x^2}\right]
\end{aligned}$$

The result folllows from Theorem 1 and the last congruence.

Corollary 2 (Catalan Numbers): Let $a(n)$ be the constant term of $(1-x)\left(2+x+\dfrac{1}{x}\right)^n$ and let

$$A(r;p) := \left(\sum_{n=0}^{rp-1} a(n)\right) \bmod p$$

Then, $A(1;p) = \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -2, & \text{if } p \equiv 2 \mod 3 \end{cases}$.

Proof:

$$\begin{aligned}
\sum_{n=0}^{p-1} a(n) &= \sum_{n=0}^{p-1} CT_x\left[(1-x)\left(2+x+\frac{1}{x}\right)^n\right] \\
&= CT_x\left[\frac{(1-x)\left(\left(2+x+\frac{1}{x}\right)^p-1\right)}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p CT_x\left[\frac{(1-x)\left(\left(2+x^p+\frac{1}{x^p}\right)-1\right)}{2+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1-x}{1+x+x^2}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1}{1+x+x^2}\right] - COEFF_{x^{p-2}}\left[\frac{1}{1+x+x^2}\right]
\end{aligned}$$

The result follows from Theorem 1 and the last congruence.

Theorem 2 (Motzkin Numbers): Let $a(n)$ be the constant term of $(1-x^2)\left(1+x+\frac{1}{x}\right)^n$ and let

$$A(r;p) := \sum_{n=0}^{rp-1} a(n) \ .$$

Then, $A(1;p) = \begin{cases} 2, & \text{if } p \equiv 1 \mod 4 \\ -2, & \text{if } p \equiv 3 \mod 4 \end{cases}$.

Proof:

$$\begin{aligned}
\sum_{n=0}^{p-1} a(n) &= \sum_{n=0}^{p-1} CT\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right] \\
&= CT\left[\frac{(1-x^2)\left(\left(1+x+\frac{1}{x}\right)^p-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1-x^2}{1+x^2}\right] \\
&\equiv_p COEFF_{x^{p-1}}\left[\frac{1}{1+x^2}\right] - COEFF_{x^{p-3}}\left[\frac{1}{1+x^2}\right]
\end{aligned}$$

The result follows from series expansion of $\frac{1}{1+x^2}$ and the last congruence.

Corollary 3: Let $a(n)$ be the constant term of $(1-x^2)\left(1+x+\dfrac{1}{x}\right)^n$ and let

$$A(r;p) := \sum_{n=0}^{rp-1} a(n) \ .$$

Then, $A(1;p) = \begin{cases} 4, & \text{if } p \equiv 1 \mod 4 \\ -4, & \text{if } p \equiv 3 \mod 4 \end{cases}$.

Proof:

$$\begin{aligned}
\sum_{n=0}^{2p-1} a(n) &= \sum_{n=0}^{2p-1} CT_x\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right] \\
&= CT_x\left[\frac{(1-x^2)\left(\left(1+x+\frac{1}{x}\right)^{2p}-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&= CT_x\left[\frac{(1-x^2)\left(\left(3+2x+x^2+\frac{2}{x}+\frac{1}{x^2}\right)^p-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&= CT_x\left[\frac{(1-x^2)\left(\left(3+2x^p+x^{2p}+\frac{2}{x^p}+\frac{1}{x^{2p}}\right)-1\right)}{1+x+\frac{1}{x}-1}\right] \\
&\equiv_p COEFF_{x^{2p-1}}\left[\frac{(1-x^2)(1+2x^p)}{1+x^2}\right] \\
&\equiv_p COEFF_{x^{2p-1}}\left[\frac{1}{1+x^2}\right] + 2COEFF_{x^{p-1}}\left[\frac{1}{1+x^2}\right] \\
&- COEFF_{x^{2p-3}}\left[\frac{1}{1+x^2}\right] - 2COEFF_{x^{p-3}}\left[\frac{1}{1+x^2}\right] .
\end{aligned}$$

The result follows from Theorem 2 and the last congruence.

Theorem 3: Let $P(x)$ be a Laurent polynomial in x and let p be a prime. Let $R(x)$ the denominator after simplifying the expression

$$\frac{P(x^p)-1}{P(x)-1}.$$

Then, for any positive integer r and Laurent polynomial $Q(x)$,

$$\left(\sum_{n=0}^{rp-1} CT_x\left[P(x)^n Q(x)\right]\right) \bmod p$$

is congruent to a p-finite sequence and can be expressed as linear combinations of finite terms of the coefficients of $\frac{1}{R(x)}$.

Theorem 4: Let p be a prime number and let r be a positive integer. Let

$$A(r,s;p) := \sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1}\binom{m+n}{m} \quad \bmod p \ .$$

Then,

$$A(r,s;p^{\alpha}) = \binom{r+s}{s} - 1 \text{ , where } \ \alpha = 1,2,3 \ .$$

Proof: Let $P(x) = 1 + \dfrac{1}{x}$ and $Q(x) = 1 + x$. Then,

$$\begin{aligned}
\sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1}\binom{m+n}{m} &= \sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1} CT_x\left[P(x)^m Q(x)^n\right] \\
&= \sum_{n=0}^{sp-1} CT_x\left[\frac{P(x)^{rp}-1)Q(x)^n}{P(x)-1}\right] \\
&= CT_x\left[\left(\frac{P(x)^{rp}-1}{P(x)-1}\right)\left(\frac{Q(x)^{sp}-1}{Q(x)-1}\right)\right] \\
&= CT_x\left[\left(\left(1+\frac{1}{x}\right)^{rp}-1\right)\left((1+x)^{sp}-1\right)\right] \\
&= CT_x\left[\left(\left(1+\frac{1}{x}\right)^{rp}-1\right)\left((1+x)^{sp}-1\right)\right] \\
&= CT_X\left[\left(1+\frac{1}{x}\right)^{rp}(1+x)^{sp} - \left(1+\frac{1}{x}\right)^{rp} - (1+x)^{sp} + 1\right] \\
&= CT_X\left[\frac{(1+x)^{rp+sp}}{x^{sp}} - 1 - 1 + 1\right] \\
&= \binom{p(r+s)}{ps} - 1 \\
&\equiv_{p^{\alpha}} \binom{r+s}{s} - 1, \text{where } \ \alpha = 1,2,3 \ .
\end{aligned}$$

The last equality follows from the beautiful result that $\binom{pa}{pb} \equiv_{p^{\alpha}} \binom{a}{b}$ for $\alpha = 1,2,3$.

Theorem 5: Let p be a prime number and let r be a positive integer. Let

$$A(r,s;p) := \left(\sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 \right) \mod p \ .$$

Then, $A(1,1;p) = \begin{cases} 0, & \text{if } p \equiv 0 \mod 3 \\ 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3 \ . \end{cases}$

Furthermore, $A(2,2;p) \equiv_p COEFF_{[x^{2p-1}y^{2p-1}]} \dfrac{1+4x^p+4y^p+16x^py^p}{(1+x+xy)(1+y+xy)}$.

Proof: Let $P(x,y) = (1+y)\left(1+\dfrac{1}{x}\right)$ and $Q(x,y) = (1+x)\left(1+\dfrac{1}{y}\right)$. First observe that

$$\binom{n+m}{m}^2 = \binom{n+m}{m}\binom{n+m}{n} = CT_{xy}\left(P(x,y)^n Q(x,y)^m\right) \ .$$

Then,

$$\begin{aligned} \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \binom{m+n}{m}^2 &= \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} CT_{xy}\left[P(x,y)^n Q(x,y)^m\right] \\ &= \sum_{m=0}^{p-1} CT_{xy}\left[\frac{P(x,y)^p-1)Q(x,y)^m}{P(x,y)-1}\right] \\ &= CT_{xy}\left[\left(\frac{P(x,y)^p-1}{P(x,y)-1}\right)\left(\frac{Q(x,y)^p-1}{Q(x,y)-1}\right)\right] \end{aligned}$$

Using the fact that $(a+b)^p \equiv_p a^p + b^p$, we get

$$\begin{aligned} \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \binom{m+n}{m}^2 &\equiv_p CT_{xy}\left[\left(\frac{P(x^p,y^p)-1}{P(x,y)-1}\right)\left(\frac{Q(x^p,y^p)-1}{Q(x,y)-1}\right)\right] \\ &\equiv_p CT_{xy}\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)x^{p-1}y^{p-1}}\right] \\ &\equiv_p COEFF[x^{p-1}y^{p-1}]\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)}\right] \\ &\equiv_p COEFF[x^{p-1}y^{p-1}]\left[\frac{1}{(1+y+xy)(1+x+xy)}\right] \end{aligned}$$

IUsing the Almkvist-Apagodu-Zeilberger algorithm, the diagonal coefficients satisfy the recurrence $N^2+N+1=0$ with the initial conditions $a(0)=1, a(1)=1, a(2)=0$. The result now follows from the fact that the recurrence is equivalent to $N^3-1=0$.

Finally, if we apply the above method to $A(2,2;p)$ we get,

$$A(2,2;p) \equiv_p COEFF_{[x^{2p-1}y^{2p-1}]} \frac{1+4x^p+4y^p+16x^py^p}{(1+x+xy)(1+y+xy)} \ .$$

Expanding, we get

$$COEFF_{[x^{2p-1}y^{2p-1}]} \frac{1}{(1+x+xy)(1+y+xy)} + 4COEFF_{[x^{p-1}y^{2p-1}]} \frac{1}{(1+x+xy)(1+y+xy)}$$
$$+4COEFF_{[x^{2p-1}y^{p-1}]} \frac{1}{(1+x+xy)(1+y+xy)} + 16COEFF_{[x^{p-1}y^{p-1}]} \frac{1}{(1+x+xy)(1+y+xy)}$$

By symmetry in the second and third term, we have

$$A(2,2;p) \equiv_p COEFF_{[x^{2p-1}y^{2p-1}]} \frac{1}{(1+x+xy)(1+y+xy)} +$$
$$8COEFF_{[x^{p-1}y^{2p-1}]} \frac{1}{(1+x+xy)(1+y+xy)} + 16COEFF_{[x^{p-1}y^{p-1}]} \frac{1}{(1+x+xy)(1+y+xy)}$$

Let p be a prime number. Then,

$$C(r,s;p) := \left(\sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^3 \right) \mod p \ .$$

Since

$$\binom{n+m}{m}^3 = CT_{xyz} \left[\frac{(1+x)^{m+n}}{x^m} \frac{(1+y)^{m+n}}{y^m} \frac{(1+z)^{m+n}}{z^n} \right] \ .$$

If we let $P(x,y,z) = \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)(1+z)$ and $Q(x,y,z) = (1+x)(1+y)\left(1+\frac{1}{z}\right)$, then we have

$$\binom{n+m}{m}^3 = CT_{xyz} \left((P(x,y)^m Q(x,y)^n \right) \ .$$

Repeating the above arguement verbatim, we arive at

$$C(1,1,;p) \equiv_p COEFF[x^{p-1}y^{p-1}z^{p-1}] \left[\frac{1}{p(x,y,z)q(x,y,z)} \right],$$

where $p(x,y,z) = 1+x+y+xy+xz+yz+xyz$ and $q(x,y,z) = 1+y+z+xy+xz+yz+xyz$.

Theorem 6: Let p be a prime number and let

$$A(r,s,t;p) := \sum_{m_1=0}^{rp-1}\sum_{m_2=0}^{sp-1}\sum_{m_3=0}^{tp-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} .$$

Then, $A(1,1,1;p) \equiv_p 1$ for $p > 2$.

Proof ($\alpha = 1$ case): Observe that

$$\binom{m_1+m_2+m_3}{m_1,m_2,m_3} = CT_{xyz}\left[\frac{(1+x+y+z)^{m_1+m_2+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right]$$

As in the proof of Theorem 2 using the fact that $(x+y)^p \equiv_p x^p + y^p$, we get

$$\begin{aligned}\sum_{m1=0}^{p-1}\sum_{m2=0}^{p-1}\sum_{m_3=0}^{p-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} &= \sum_{m_1=0}^{p-1}\sum_{m_2=0}^{p-1}\sum_{m_3=0}^{p-1} CT_{xyz}\left[\frac{(1+x+y+z)^{m_1+m_1+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right] \\ &\equiv_p CT_{xyz}\frac{(1+x^py^p+x^pz^p)(1+y^px^p+y^pz^p)(1+z^px^p+z^py^p)}{(xyz)^{p-1}(1+x+y)(1+x+z)(1+y+z)} \\ &\equiv_p COEFF_{[x^{p-1}y^{p-1}z^{p-1}]}\frac{1}{(1+x+y)(1+x+z)(1+y+z)}\end{aligned}$$

Using Almkvist-Apagodu-Zeilberger algorithm, the diagonal coefficients, $a(n) := a(n,n,n)$, satisfy the recurrence

$$24(5n+11)(3n+5)(3n+4)a(n) + (295n^3+1614n^2+2855n+1620)a(n+1)$$

$$+2(2n+5)(n+2)(5n+6)a(n+2) = 0 .$$

Passing to mod p, the terms with powers of p are zero and the recurrence reduces to the much simpler second order recurrence,

$$288a(p) + 84a(p+1) + 6a(p+2) = 0$$

with initial conditions $a(0) = 1$ and $a(1) = -14$, whose solution is given by

$$a(p) \equiv_p 4(-8)^p - 3(-6)^p .$$

From the above equation, we know that $A(1,1,1;p) \equiv a(p-1)$ (mod p). Thus,

$$A(1,1,1;p) = a(p-1) \equiv_p 4(-8)^{p-1} - 3(-6)^{p-1} \equiv_p 2^{p-1}, .$$

.
Now it suffices to show that $2^{p-1} - 1 \equiv_p 0$. But this follows fom

$$2^{p-1} - 1 = \frac{1}{2}(2^p - 2) = \frac{1}{2}((1+1)^p - 2) \equiv_p \frac{1}{2}(1+1-2) = 0 .$$

Observation 1: $A(1,1,1;p) \equiv_{p^\alpha} 1$ for $p > 2$, for $\alpha = 2,3$.

Observation 2: The above computer data suggests that $A(2,2,2;p) \equiv_{p^\alpha} 16$ for $p \geq 17$, for $\alpha = 1,2,3$

In general, if we denote,

$$A(r_1, r_2, \ldots, r_a; p) := \sum_{m_1=0}^{r_1 p-1} \sum_{m_2=0}^{r_2 p-1} \cdots \sum_{m_a=0}^{r_a p-1} \binom{m_1 + m_2 \ldots + m_a}{m_1, m_2, \ldots, m_a}.$$

Then,

(1) $A(1,1,\ldots,1;p) \equiv_p Coeff_{[x_1^{p-1} x_2^{p-1} \ldots x_a^{p-1}]} \dfrac{1}{\prod_{j=1}^a (1 + \sum_{i=1, i\neq j}^a x_i)}$.

(2) **Conjecture**: $A(1,1,\ldots,1;p) \equiv_p 1$ for $p > 2$ and $A(1,1,\ldots,1;2) \equiv_p 0$.

Proof (of i) : Observe that

$$\binom{m_1 + m_2 \ldots + m_a}{m_1, m_2, \ldots, m_a} = CT_{x_1 x_2 \ldots x_a} \left[\frac{(1 + x_1 + x_2 + \ldots + x_a)^{m_1 + m_2 + \ldots + m_a}}{\prod_{i=1}^a x_i^{m_i}} \right]$$

As in the proof of Theorem 4,

$$\begin{aligned} \sum_{m_1=0}^{p-1} \cdots \sum_{m_a=0}^{p-1} \binom{m_1 + m_2 \ldots + m_a}{m_1, m_2, \ldots, m_a} &= \sum_{m_1=0}^{p-1} \cdots \sum_{m_a=0}^{p-1} CT_{x_1 x_2 \ldots x_a} \left[\frac{(1 + x_1 + x_2 + \ldots + x_a)^{\sum_{j=1}^a m_j}}{\prod_{i=1}^a x_i^{m_i}} \right] \\ &\equiv_p COEFF_{[x_1^{p-1} x_2^{p-1} \ldots x_a^{p-1}]} \frac{1}{\prod_{j=1}^a \left(1 + \sum_{i=1, i\neq j}^a x_i\right)}. \end{aligned}$$

Apery's Number:

Apery's number is defined by

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and it is known that

$$A(n) = CT_{xyz}\left[\left(\frac{(1+x)(1+y)(1+z)(1+y+z+yz+xyz)}{xyz}\right)^n\right]$$

Let p be a prime number and define

$$a(p) := \sum_{n=0}^{p-1} A(n) \ .$$

We can express $a(p)$ as

$$\begin{aligned}
\sum_{n=0}^{p-1}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 &= \sum_{n=0}^{p-1} CT_{xyz}\left[\left(\frac{(1+x)(1+y)(1+z)(1+y+z+yz+xyz)}{xyz}\right)^n\right] \\
&= CT_{xyz}\left[R1(x,y,z) + R2(x,y,z)\right] \\
&= CT_{xyz}\left[\frac{R1(x,y,z)}{x^{p-1}y^{p-1}z^{p-1}}\right] \\
&= CT_{xyz}\left[\frac{P(x,y,z)}{x^{p-1}y^{p-1}z^{p-1}Q(x,y,z)}\right] \\
&\equiv_p COEFF_{x^{p-1}y^{p-1}z^{p-1}} \frac{P(x^p,y^p,z^p)}{Q(x,y,z)} \\
&\equiv_p COEFF_{x^{p-1}y^{p-1}z^{p-1}} \left[\frac{1}{Q(x,y,z)}\right]
\end{aligned}$$

where, $R2(x,y,z) = \frac{xyz}{Q(x,y,z)}$ and has zero constant term and $R1(x,y,z) = \frac{P(x,y,z)}{Q(x,y,z)}$, where

$$P(x,y,z) = (x+1)^p(y+1)^p(1+z)^p((1+y+z+yz+xyz)^p$$

and

$$\begin{aligned}
Q(x,y,z) = {} & x^2y^2z^2 + x^2y^2z + x^2yz^2 + 2xy^2z^2 + x^2yz + 3xy^2z + 3xyz^2 + y^2z^2 + xy^2 \\
& +4xyz + xz^2 + 2y^2z + 2yz^2 + 2xy + 2xz + y^2 + 4yz + z^2 + x + 2y + 2z + 1 \ .
\end{aligned}$$

Therefore,

$$a(p) \equiv \alpha(p-1,p-1,p-1) (\text{mod } p)$$

where

$$Q(x,y,z) = \sum_{m,n,k} \alpha(m,n,k) x^n y^m z^k \ .$$

Toward Z-W Sun's Conjecture:

Z-W Sun conjectures that

$$\sum_{n=0}^{p-1}\sum_{k=0}^{n}(-1)^{n+k}8^k\binom{n}{k}^3 \equiv \left(\frac{p}{3}\right) \pmod{p^2} \ .$$

Using the fact that

$$\binom{n}{k}^3 = CT_{xyz}\left[\frac{(1+x)^n(1+y)^n(1+z)^n}{x^k y^k z^k}\right] \ .$$

we get

$$\begin{aligned}
\sum_{n=0}^{p-1}\sum_{k=0}^{n}(-1)^{n+k}8^k\binom{n}{k}^3 &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}(-1)^{n+k}8^k CT_{xyz}\left[\frac{(1+x)^n(1+y)^n(1+z)^n}{x^k y^k z^k}\right] \\
&= CT_{xyz}\left[R1(x,y,z)+R2(x,y,z)+R3(x,y,z)+R4(x,y,z)\right]
\end{aligned}$$

where

$$R1(x,y,z) = -\frac{(-1)^p((x+1)(y+1)(1+z))^p xyz}{(xyz+xy+xz+yz+x+y+z+2)(xyz+8)}$$

$$R2(x,y,z) = \frac{(-8)^{p+1}((x+1)(y+1)(1+z))^p}{x^{p-1}y^{p-1}z^{p-1}(7xyz+8xy+8xz+8yz+8x+8y+8z+8)(xyz+8)}$$

$$R3(x,y,z) = \frac{xyz}{(xyz+xy+xz+yz+x+y+z+2)(xyz+8)}$$

and

$$R4(x,y,z) = -\frac{8xyz}{(7xyz+8xy+8xz+8yz+8x+8y+8z+8)(xyz+8)} \ .$$

Since the constant terms of $R1(x,y,z), R3(x,y,z)$ and $R4(x,y,z)$ are zero, we get,

$$\begin{aligned}
\sum_{n=0}^{p-1}\sum_{k=0}^{n}(-1)^{n+k}8^k\binom{n}{k}^3 &= COEFF_{x^{p-1}y^{p-1}z^{p-1}}\left[\frac{(-8)^{p+1}((1+x)(1+y)(1+z))^p}{(7xyz+8(xy+xz+yz+x+y+z+1))(xyz+8)}\right] \\
&\equiv_p COEFF_{x^{p-1}y^{p-1}z^{p-1}}\left[\frac{(-8)^{p+1}}{(7xyz+8(xy+xz+yz+x+y+z+1))(xyz+8)}\right]
\end{aligned}$$

Theorem 7: Let $P(x,y,z)$ and $Q(x,y,z)$ be Laurent polynomials in x,y and z. Let $T(x,y,z)$ be the denominator (after clearing!) of

$$\frac{(P(x^p,y^p,z^p)-1)(Q(x^p,y^p,z^p)-1)}{(P(x,y,z)-1)(Q(x,y,z)-1)} \, .$$

Then, for a prime p, Laurent polynomial $R(x,y,z)$, and positive integers r,s,t,

$$\left(\sum_{k=0}^{rp-1} \sum_{n=0}^{sp-1} \sum_{n=0}^{tp-1} CT_{xyz} \left(P(x,y,z)^n Q(x,y,z)^k R(x,y,z) \right) \right) \mod p$$

is congruent to a P-finite sequence and can be expressed as a finite linear combination of the coefficients of $\dfrac{1}{T(x,y,z)}$.