Odd-Rule Cellular Automata on the Square Grid

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Abstract

An "odd-rule" cellular automaton (CA) is defined by specifying a neighborhood for each cell, with the rule that a cell turns ON if it is in the neighborhood of an odd number of ON cells at the previous generation, and otherwise turns OFF. We classify all the odd-rule CAs defined by neighborhoods which are subsets of a 3×3 grid of square cells. There are 86 different CAs modulo trivial symmetries. When we consider only the different sequences giving the number of ON cells after n generations, the number drops to 48, two of which are the Moore and von Neumann CAs. This classification is carried out by using the "meta-algorithm" described in an earlier paper to derive the generating functions for the 86 sequences, and then removing duplicates. The fastest-growing of these CAs is neither the Fredkin nor von Neumann neighborhoods, but rather one defined by "Odd-rule" 365, which turns ON almost 75% of all possible cells.

1 Introduction

As in [1, 2, 10], our goal is to study how fast activity spreads in cellular automata (CAs): more precisely, if we start with a single ON cell, how many cells will be ON after n generations? For additional background see [4, 5, 6, 7, 9, 11, 12, 13, 14].

Continuing the investigations begun in [2, 10], we consider "odd-rule" CAs, concentrating on the two-dimensional rules defined by neighborhoods that are subsets of a 3×3 square grid. This family of CAs includes two that were the main subject of [10], namely Fredkin's Replicator, which is based on the Moore neighborhood, and another which is based on the von Neumann neighborhood

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with a center cell. One of the goals of the present paper is to use the meta-algorithm from our paper [2] to obtain generating functions, with proofs, for all these sequences. This provides alternative (computer-generated) proofs of Theorems 4 and 5 of [10]. Another member of this family is the CA defined by the one-dimensional Rule 150 in the Wolfram numbering scheme [9, 13, 14].

The Wolfram numbering scheme is not, however, particularly convenient for dealing with these 3×3 neighborhoods, and in §3 we introduce a simpler numbering scheme based on reading the neighborhood as a triple of octal numbers.

Section 2 gives the definitions of an odd-rule cellular automaton and the run length transform, and quotes two essential theorems from [10]. Section 3 classifies odd-rule CAs that are defined by neighborhoods that are subsets of the 3×3 grid: if we ignore trivial differences there are 86 different CAs (Theorem 3), shown in Figs. 1, 2, 3 and Tables 1, 2, 3. In Section 4 we define two CAs to be "combinatorially equivalent" if the numbers of ON cells after n generations are the same for all n. Up to combinatorial equivalence there are 48 different CAs (Theorem 4), shown in Tables 1, 2, 3. In Section 5 we discuss three further topics: which CA has the greatest growth rate (§5.1 – the answer is unexpected), which has the slowest growth rate (§5.2), and the question of explaining why certain pairs of CAs turn out to have the same generating function (§5.3). The 48 distinct generating functions are given in an Appendix.

2 Odd-rule CAs

We consider cellular automata whose cells form the 2-dimensional square lattice $\mathbb{Z}^2$. Each cell is either ON or OFF, and an ON cell with center at the lattice point $(i, j) \in \mathbb{Z}^2$ will be identified with the monomial $x^i y^j$, which we regard as an element of the ring of Laurent polynomials $\mathcal{R} := \mathrm{GF}(2)[x, x^{-1}, y, y^{-1}]$ with mod 2 coefficients. The state of the CA is specified by giving the formal sum S of all its ON cells. As long as only finitely many cells are ON, S is indeed an element of $\mathcal{R}$.

An "odd-rule" CA (this name was introduced in [10], although of course the concept has been known for as long as people have been studying CAs) is defined by specifying a neighborhood of the cell at the origin, given by an element $F \in \mathcal{R}$ listing the cells in the neighborhood. A typical example is the Moore neighborhood, which consists of the eight cells surrounding the cell at the origin (see Odd-rule 757 in Fig. 3), and is specified by

$$\begin{aligned} F &:= \frac{1}{xy} + \frac{1}{y} + \frac{x}{y} + \frac{1}{x} + x + \frac{y}{x} + y + xy \\ &= \left(\frac{1}{x} + 1 + x\right)\left(\frac{1}{y} + 1 + y\right) - 1 \quad \in \mathcal{R} \end{aligned} \tag{1}$$

The neighborhood of an arbitrary cell $x^r y^s$ is obtained by shifting F so it is centered at that cell, that is, by the product $x^r y^s F \in \mathcal{R}$. Given F, the corresponding *odd-rule* CA is defined by the rule that a cell $x^r y^s$ is ON at generation $n + 1$ if it is the neighbor of an odd number of cells that were ON at generation n, and is otherwise OFF.

Our goal is to find $a_n(F)$, the number of ON cells at the nth generation when the CA is started in generation 0 with a single ON cell at the origin. For odd-rule CAs there is a simple formula for $a_n(F)$. The number of nonzero terms in an element $P \in \mathcal{R}$ will be denoted by $|P|$.

Theorem 1. *[10] For an odd-rule CA with neighborhood F, the state at generation n is equal to F^n, and $a_n(F) = |F^n|$.*

The sequences $[a_n(F), n \geq 0]$ are most easily described using the "run length transform", an operation on number sequences also introduced in [10]. For an integer $n \geq 0$, let $\mathcal{L}(n)$ denote the list of the lengths of the maximal runs of 1s in the binary expansion of n. For example, since the binary expansion of 55 is 110111, containing runs of 1s of lengths 2 and 3, $\mathcal{L}(55) = [2, 3]$. $\mathcal{L}(0)$ is the empty list, and $\mathcal{L}(n)$ for $n = 1, \ldots, 12$ is respectively $[1], [1], [2], [1], [1, 1], [2], [3], [1], [1, 1], [1, 1], [1, 2], [2]$ (A245562[2]).

Definition. The *run length transform* of a sequence $[S_n, n \geq 0]$ is the sequence $[T_n, n \geq 0]$ given by

$$T_n = \prod_{i \in \mathcal{L}(n)} S_i. \tag{2}$$

Note that T_n depends only on the lengths of the runs of 1s in the binary expansion of n, not on the order in which they appear. For example, since $\mathcal{L}(11) = [1, 2]$ and $\mathcal{L}(13) = [2, 1]$, $T_{11} = T_{13} = S_1 S_2$. Also $T_0 = 1$ (the empty product), so the value of S_0 is never used, and will usually be taken to be 1. For further properties and additional examples of the run length transform see [10]. See especially [10, Table 4], which shows how the transformed sequence has a natural division into blocks of successive lengths $1, 1, 2, 4, 8, 16, 32, \ldots$.

Define the *height* $\mathrm{ht}(F)$ of an element $F \in \mathcal{R}$ to be $\max\{|i|, |j|\}$ for any monomial $x^i y^j$ in F. If $\mathrm{ht}(F) = h$, the cells in F are a subset of the cells in a $(2h+1) \times (2h+1)$ array of squares centered at the origin. In particular, if $\mathrm{ht}(F) \leq 1$, we have the following:

Theorem 2. *[10] If the neighborhood F is a subset of the 3×3 grid of cells centered at the origin, then $[a_n(F), n \geq 0]$ is the run length transform of the subsequence $[b_n(F), n \geq 0]$, where $b_n(F) := a_{2^n - 1}(F)$.*

3 Trivially equivalent neighborhoods

From now on we assume that F has height at most one, i.e., is a subset of the 3×3 grid of cells centered at the origin. In view of Theorem 1, $a_n(F)$ is unchanged if we multiply (or divide) F by x or y (these operations simply translate the configuration of ON states in the (x, y)-plane).

We can also apply any of the eight symmetries of the square (rotations and/or reflections, forming the dihedral group of order eight) to F without changing $a_n(F)$.

We therefore say that two neighborhoods $F \in \mathcal{R}$, $G \in \mathcal{R}$ are *trivially* (or *affinely*) equivalent if one can be translated into the other by repeated translations, rotations, and reflections.

Theorem 3. *Up to trivial equivalence, there are* 86 *distinct height-one neighborhoods, as shown in Figs. 1, 2, 3, and again in Tables 1, 2, 3.*

Proof. Hand calculation, followed by computer verification. □

Rather than use the Wolfram numbering scheme, which here would involve numbers as large as 2^{512}, we describe the neighborhood F by a three-digit octal number, the "Odd-rule" number, obtained by reading the ON cells in the 3×3 grid from left to right, top to bottom.

The *canonical* Odd-rule number for F is then the smallest of the Odd-rule numbers associated with any neighborhood that is trivially equivalent to F.

For example, the neighborhood $F = 1 + x \in \mathcal{R}$, consisting of two adjacent cells, can be shifted or rotated into 12 different positions, described by the octal numbers 600, 300, 060, 030, 006, 003, 440, 044, 220, 022, 110, 011. The smallest of these is 003 (corresponding to $1/y + x/y$), which is therefore the canonical Odd-rule number for this F (see the the third figure in Fig. 1).

[2] Six-digit numbers prefixed by A refer to entries in [8].

Figure 1: Up to trivial equivalence, there are 86 distinct height-one neighborhoods, shown in Figs. 1, 2, 3 together with their canonical Odd-rule numbers.

The Odd-rule number for Wolfram's one-dimensional Rule 150 is 007. The two CAs that were the main subject of [10], namely "Fredkin's Replicator", which is based on the Moore neighborhood, and the CA based on the von Neumann neighborhood with a central cell, are Odd-rules 757 and 272, respectively. The von Neumann neighborhood without the central cell is Odd-rule 252, and the full 3×3 neighborhood is Odd-rule 777.

The canonical Odd-rule numbers for all 86 trivially inequivalent height-one neighborhoods are shown in Figs. 1, 2, 3, which gives graphical representations of the neighborhoods. These 86 neighborhoods are also shown in Tables 1, 2, 3. The first column of these tables gives the canonical Odd-rule number, the second column gives the number of cells in the neighborhood, the third column gives the binary representation of the neighborhood, and the fourth column gives the corresponding Laurent polynomial F.

4 Combinatorially equivalent neighborhoods

Since we are mostly interested in the sequences that give the number of ON cells after n generations, we shall say that two height-one neighborhoods F and G are *combinatorially equivalent* if $a_n(F) = a_n(G)$ for all $n \geq 0$. In view of Theorem 2, an equivalent condition is that $b_n(F) = b_n(G)$ for all $n \geq 0$.

Figure 2: See caption to Fig. 1.

Theorem 4. *Up to combinatorial equivalence, there are* 48 *distinct height-one neighborhoods.*

Proof. Using the Maple programs `ARLT` and `GFsP` described in [2] (available from [3]), we computed the generating function for the $b_n(F)$ sequence corresponding to each of the 86 neighborhoods listed in Theorem 3. After removing duplicates, 48 remained (see the Appendix). $\square$

As representative for each equivalence class of neighborhoods we take the one with the smallest Odd-rule number. These 48 combinatorially inequivalent neighborhoods can be seen in Tables 1, 2, 3, where they are distinguished by having the sequence numbers in [8] for the $a_n(F)$ and $b_n(F)$ sequences in the final column of the tables. If the $a_n(F)$ and $b_n(F)$ sequences are the same as those for some earlier rule, this is also indicated in the final column.

The generating functions for the 48 $b_n(F)$ sequences, together with the corresponding sequence numbers, are given in the Appendix. They are shown in such a way that they can be easily copied into a computer algebra system (that is, they are given in a linear rather than two-dimensional format).

In particular, the generating functions for the Fredkin and von Neumann-with-center CAs match those derived in [10], and so provide an alternative proof for Theorems 4 and 5 of that paper.

Figure 3: See caption to Fig. 1.

5 Further topics

5.1 The highest growth rate.

It is natural to ask which rule produces the greatest number of ON cells. We just consider the number that are ON at generation $2^n - 1$, that is, the sequence $b_n(F) = a_{2^n-1}(F)$, since by Theorem 2 these are local maxima, and all other values of $a_m(F)$ are products of these values.

The most fecund rule is somewhat of a surprise: it is Odd-rule 365, seen in the top left figure in Fig. 3. This is the unique winner, well ahead of the more obvious candidates such as rules 252, 272, 525, 757, or 777.

For Odd-rule 365 we have $F = 1/(xy) + 1/x + x/y + 1 + y + xy$, $b_n(F) = 3.4^n - 2.3^n, n \geq 0$, with generating function $(1-x)/((1-3x)(1-4x))$, recurrence $b_{n+1} = 7b_n - 12b_{n-1}$, and initial values

1, 6, 30, 138, 606, 2586, 10830, 44778, 183486, 747066, 3027630, ... (A255463)

Other rules do better at the start, but for $n \geq 4$ Odd-rule 365 is the winner, and thus, for any height-one Odd-rule neighborhood F,

$$b_n(F) \;\leq\; 3.4^n \;-\; 2.3^n, \quad n \geq 4. \tag{3}$$

Equality holds if and only if F is trivially equivalent to Odd-rule 365.

After $2^n - 1$ generations of any odd-rule height-one CA that starts with a single ON cell at generation 0, the ON cells are contained in the square of side $2^{n+1} - 1$ centered at the origin. Odd-rule 365 turns ON a fraction

$$\frac{3.4^n - 2.3^n}{(2^{n+1}-1)^2} \tag{4}$$

of these, which approaches 3/4 as $n \mapsto \infty$.

Figure 4 shows generation 15 of this CA, containing $a_{15}(F) = b_4(F) = 606$ ON cells, and Fig. 6 (to be read from right to left, top to bottom) shows the evolution of this automaton up to this point. The ON cells are colored black.

From Theorem 3 of [10], the $a_n(F)$ sequence, which has initial terms

1, 6, 6, 30, 6, 36, 30, 138, 6, 36, 36, 180, 30, 180, 138, 606, 6, 36, 36, 180, ... A255462 ,

satisfies the recurrence $a_{2t} = a_t$, $a_{4t+1} = 6a_t$, $a_{4t+3} = 7a_{2t+1} - 12a_t$ for $t > 0$, with $a_0 = 1$.

Incidentally, the runner-up is Odd-rule 537, for which the fraction of ON cells at generations $2^n - 1$ approaches 2/3.

5.2 The lowest growth rate.

Odd-rules 000, 001, 003, 007 have b_n equal to 0, 1, 2^n, and $(2^{n+2} - (-1)^n)/3$, respectively. But the slowest-growing properly two-dimensional rule is Odd-rule 013, for which $b_n = 3^n$. Figure 5 (drawn at the same scale as Fig. 4) shows generation 15, containing a mere $a_{15} = b_4 = 81$ ON cells.

5.3 Explaining combinatorial equivalence.

In some cases it is easy to explain why two different neighborhoods have the same a_n (and b_n) sequences, i.e., are combinatorially equivalent. Let us denote combinatorial equivalence by $\sim$. All five of the trivially inequivalent two-celled neighborhoods are combinatorially equivalent—for example, rule 003, $1/y + x/y \sim 1 + x \sim 1/y + x$, which is rule 012. To see that the four-celled rules 033 $(1 + x + 1/y + x/y)$ and 505 $(y/x + x?y + 1/(xy) + x/y)$ are equivalent, replace x by x^2 in the former, then divide by x, replace y by y^2, and finally multiply by y. For other pairs, such as the seven-celled rules 376 and 557, there does not seem to be a simple proof that they are combinatorially equivalent, even though we know (by the theory developed in [2]) that this is true.

Figure 4: Odd-rule 365, the fastest-growing, after 15 generations (see also Fig 6). There are 606 ON cells.

Figure 5: Odd-rule 013, the slowest-growing, after 15 generations (on the same scale). There are 81 ON cells.

Figure 6: Generations 0 to 15 of Odd-rule 365 (to be read from right to left, top to bottom).

Table 1: Tables 1, 2, 3 show the 86 trivially inequivalent neighborhoods and the 48 combinatorially inequivalent ones. The asterisk denotes multiplication. See text for further details.

Rule	Cells	Neighborhood	F	$a_n(F), b_n(F)$
000	0	[0, 0, 0, 0, 0, 0, 0, 0, 0]	0	A000004, A000004
001	1	[0, 0, 0, 0, 0, 0, 0, 0, 1]	x/y	A000012, A000012
003	2	[0, 0, 0, 0, 0, 0, 0, 1, 1]	$1/y + x/y$	A001316, A000079
005	2	[0, 0, 0, 0, 0, 0, 1, 0, 1]	$1/(x * y) + x/y$	same as Odd-rule 003
012	2	[0, 0, 0, 0, 0, 1, 0, 1, 0]	$x + 1/y$	same as Odd-rule 003
014	2	[0, 0, 0, 0, 0, 1, 1, 0, 0]	$x + 1/(x * y)$	same as Odd-rule 003
104	2	[0, 0, 1, 0, 0, 0, 1, 0, 0]	$x * y + 1/(x * y)$	same as Odd-rule 003
007	3	[0, 0, 0, 0, 0, 0, 1, 1, 1]	$1/(x * y) + 1/y + x/y$	A071053, A001045
013	3	[0, 0, 0, 0, 0, 1, 0, 1, 1]	$x + 1/y + x/y$	A048883, A000244
015	3	[0, 0, 0, 0, 0, 1, 1, 0, 1]	$x + 1/(x * y) + x/y$	same as Odd-rule 013
016	3	[0, 0, 0, 0, 0, 1, 1, 1, 0]	$x + 1/(x * y) + 1/y$	same as Odd-rule 013
025	3	[0, 0, 0, 0, 1, 0, 1, 0, 1]	$1 + 1/(x * y) + x/y$	same as Odd-rule 013
105	3	[0, 0, 1, 0, 0, 0, 1, 0, 1]	$x * y + 1/(x * y) + x/y$	same as Odd-rule 013
106	3	[0, 0, 1, 0, 0, 0, 1, 1, 0]	$x * y + 1/(x * y) + 1/y$	same as Odd-rule 013
124	3	[0, 0, 1, 0, 1, 0, 1, 0, 0]	$1 + x * y + 1/(x * y)$	same as Odd-rule 007
141	3	[0, 0, 1, 1, 0, 0, 0, 0, 1]	$x * y + 1/x + x/y$	same as Odd-rule 013
142	3	[0, 0, 1, 1, 0, 0, 0, 1, 0]	$x * y + 1/x + 1/y$	same as Odd-rule 013
017	4	[0, 0, 0, 0, 0, 1, 1, 1, 1]	$x + 1/(x * y) + 1/y + x/y$	A253064, A087206
027	4	[0, 0, 0, 0, 1, 0, 1, 1, 1]	$1 + 1/(x * y) + 1/y + x/y$	same as Odd-rule 017
033	4	[0, 0, 0, 0, 1, 1, 0, 1, 1]	$1 + x + 1/y + x/y$	A102376, A000302
035	4	[0, 0, 0, 0, 1, 1, 1, 0, 1]	$1 + x + 1/(x * y) + x/y$	A255297, A027649
036	4	[0, 0, 0, 0, 1, 1, 1, 1, 0]	$1 + x + 1/(x * y) + 1/y$	same as Odd-rule 033
055	4	[0, 0, 0, 1, 0, 1, 1, 0, 1]	$1/x + x + 1/(x * y) + x/y$	same as Odd-rule 033
107	4	[0, 0, 1, 0, 0, 0, 1, 1, 1]	$x * y + 1/(x * y) + 1/y + x/y$	same as Odd-rule 017
116	4	[0, 0, 1, 0, 0, 1, 1, 1, 0]	$x * y + x + 1/(x * y) + 1/y$	same as Odd-rule 035
125	4	[0, 0, 1, 0, 1, 0, 1, 0, 1]	$1 + x * y + 1/(x * y) + x/y$	same as Odd-rule 017
126	4	[0, 0, 1, 0, 1, 0, 1, 1, 0]	$1 + x * y + 1/(x * y) + 1/y$	same as Odd-rule 017
143	4	[0, 0, 1, 1, 0, 0, 0, 1, 1]	$x * y + 1/x + 1/y + x/y$	A255298, A255299
145	4	[0, 0, 1, 1, 0, 0, 1, 0, 1]	$x * y + 1/x + 1/(x * y) + x/y$	same as Odd-rule 035
146	4	[0, 0, 1, 1, 0, 0, 1, 1, 0]	$x * y + 1/x + 1/(x * y) + 1/y$	A255302, A255303
151	4	[0, 0, 1, 1, 0, 1, 0, 0, 1]	$x * y + 1/x + x + x/y$	same as Odd-rule 017
152	4	[0, 0, 1, 1, 0, 1, 0, 1, 0]	$x * y + 1/x + x + 1/y$	same as Odd-rule 146
154	4	[0, 0, 1, 1, 0, 1, 1, 0, 0]	$x * y + 1/x + x + 1/(x * y)$	same as Odd-rule 033
161	4	[0, 0, 1, 1, 1, 0, 0, 0, 1]	$1 + x * y + 1/x + x/y$	A255300, A255301
162	4	[0, 0, 1, 1, 1, 0, 0, 1, 0]	$1 + x * y + 1/x + 1/y$	same as Odd-rule 033
252	4	[0, 1, 0, 1, 0, 1, 0, 1, 0]	$y + 1/x + x + 1/y$	same as Odd-rule 033
505	4	[1, 0, 1, 0, 0, 0, 1, 0, 1]	$y/x + x * y + 1/(x * y) + x/y$	same as Odd-rule 033

Table 2: Tables 1, 2, 3 show the 86 trivially inequivalent neighborhoods and the 48 combinatorially inequivalent ones. See text for further details.

Rule	Cells	Neighborhood	F	$a_n(F), b_n(F)$
037	5	[0, 0, 0, 0, 1, 1, 1, 1, 1]	$1 + x + 1/(x * y) + 1/y + x/y$	A255445, A001834
057	5	[0, 0, 0, 1, 0, 1, 1, 1, 1]	$1/x + x + 1/(x * y) + 1/y + x/y$	A072272, A007483
117	5	[0, 0, 1, 0, 0, 1, 1, 1, 1]	$x * y + x + 1/(x * y) + 1/y + x/y$	A255304, A255442
127	5	[0, 0, 1, 0, 1, 0, 1, 1, 1]	$1 + x * y + 1/(x * y) + 1/y + x/y$	same as Odd-rule 117
136	5	[0, 0, 1, 0, 1, 1, 1, 1, 0]	$1 + x * y + x + 1/(x * y) + 1/y$	same as Odd-rule 037
147	5	[0, 0, 1, 1, 0, 0, 1, 1, 1]	$x * y + 1/x + 1/(x * y) + 1/y + x/y$	A255443, A255444
153	5	[0, 0, 1, 1, 0, 1, 0, 1, 1]	$x * y + 1/x + x + 1/y + x/y$	A255454, A255455
155	5	[0, 0, 1, 1, 0, 1, 1, 0, 1]	$x * y + 1/x + x + 1/(x * y) + x/y$	same as Odd-rule 037
156	5	[0, 0, 1, 1, 0, 1, 1, 1, 0]	$x * y + 1/x + x + 1/(x * y) + 1/y$	A255452, A255453
163	5	[0, 0, 1, 1, 1, 0, 0, 1, 1]	$1 + x * y + 1/x + 1/y + x/y$	A255456, A255457
165	5	[0, 0, 1, 1, 1, 0, 1, 0, 1]	$1 + x * y + 1/x + 1/(x * y) + x/y$	A255446, A255447
166	5	[0, 0, 1, 1, 1, 0, 1, 1, 0]	$1 + x * y + 1/x + 1/(x * y) + 1/y$	A255450, A255451
171	5	[0, 0, 1, 1, 1, 1, 0, 0, 1]	$1 + x * y + 1/x + x + x/y$	A253065, A253067
172	5	[0, 0, 1, 1, 1, 1, 0, 1, 0]	$1 + x * y + 1/x + x + 1/y$	same as Odd-rule 166
174	5	[0, 0, 1, 1, 1, 1, 1, 0, 0]	$1 + x * y + 1/x + x + 1/(x * y)$	same as Odd-rule 057
253	5	[0, 1, 0, 1, 0, 1, 0, 1, 1]	$y + 1/x + x + 1/y + x/y$	same as Odd-rule 156
255	5	[0, 1, 0, 1, 0, 1, 1, 0, 1]	$y + 1/x + x + 1/(x * y) + x/y$	A255458, A255459
272	5	[0, 1, 0, 1, 1, 1, 0, 1, 0]	$1 + y + 1/x + x + 1/y$	same as Odd-rule 057
345	5	[0, 1, 1, 1, 0, 0, 1, 0, 1]	$y + x * y + 1/x + 1/(x * y) + x/y$	A255448, A255449
507	5	[1, 0, 1, 0, 0, 0, 1, 1, 1]	$y/x + x * y + 1/(x * y) + 1/y + x/y$	same as Odd-rule 057
525	5	[1, 0, 1, 0, 1, 0, 1, 0, 1]	$1 + y/x + x * y + 1/(x * y) + x/y$	same as Odd-rule 057
077	6	[0, 0, 0, 1, 1, 1, 1, 1, 1]	$1 + 1/x + x + 1/(x * y) + 1/y + x/y$	A246037, A246036
137	6	[0, 0, 1, 0, 1, 1, 1, 1, 1]	$1 + x * y + x + 1/(x * y) + 1/y + x/y$	A255464, A255465
157	6	[0, 0, 1, 1, 0, 1, 1, 1, 1]	$x * y + 1/x + x + 1/(x * y) + 1/y + x/y$	A255468, A255469
167	6	[0, 0, 1, 1, 1, 0, 1, 1, 1]	$1 + x * y + 1/x + 1/(x * y) + 1/y + x/y$	A255466, A255467
173	6	[0, 0, 1, 1, 1, 1, 0, 1, 1]	$1 + x * y + 1/x + x + 1/y + x/y$	A255475, A255476
175	6	[0, 0, 1, 1, 1, 1, 1, 0, 1]	$1 + x * y + 1/x + x + 1/(x * y) + x/y$	A253069, A253070
176	6	[0, 0, 1, 1, 1, 1, 1, 1, 0]	$1 + x * y + 1/x + x + 1/(x * y) + 1/y$	A255470, A255471
257	6	[0, 1, 0, 1, 0, 1, 1, 1, 1]	$y + 1/x + x + 1/(x * y) + 1/y + x/y$	A255473, A255474
273	6	[0, 1, 0, 1, 1, 1, 0, 1, 1]	$1 + y + 1/x + x + 1/y + x/y$	same as Odd-rule 176
275	6	[0, 1, 0, 1, 1, 1, 1, 0, 1]	$1 + y + 1/x + x + 1/(x * y) + x/y$	A253066, A253068
347	6	[0, 1, 1, 1, 0, 0, 1, 1, 1]	$y + x * y + 1/x + 1/(x * y) + 1/y + x/y$	A253100, A253101
356	6	[0, 1, 1, 1, 0, 1, 1, 1, 0]	$y + x * y + 1/x + x + 1/(x * y) + 1/y$	A247640, A164908
365	6	[0, 1, 1, 1, 1, 0, 1, 0, 1]	$1 + y + x * y + 1/x + 1/(x * y) + x/y$	A255462, A255463
517	6	[1, 0, 1, 0, 0, 1, 1, 1, 1]	$y/x + x * y + x + 1/(x * y) + 1/y + x/y$	A255460, A255461
527	6	[1, 0, 1, 0, 1, 0, 1, 1, 1]	$1 + y/x + x * y + 1/(x * y) + 1/y + x/y$	A255295, A255296
555	6	[1, 0, 1, 1, 0, 1, 1, 0, 1]	$y/x + x * y + 1/x + x + 1/(x * y) + x/y$	same as Odd-rule 077

Table 3: Tables 1, 2, 3 show the 86 trivially inequivalent neighborhoods and the 48 combinatorially inequivalent ones. See text for further details.

Rule	Cells	Neighborhood	F	$a_n(F), b_n(F)$
177	7	[0, 0, 1, 1, 1, 1, 1, 1, 1]	$1 + x * y + 1/x + x + 1/(x * y)$ $+1/y + x/y$	A255277, A255278
277	7	[0, 1, 0, 1, 1, 1, 1, 1, 1]	$1 + y + 1/x + x + 1/(x * y)$ $+1/y + x/y$	A255279, A255280
357	7	[0, 1, 1, 1, 0, 1, 1, 1, 1]	$y + x * y + 1/x + x + 1/(x * y)$ $+1/y + x/y$	A253071, A253072
367	7	[0, 1, 1, 1, 1, 0, 1, 1, 1]	$1 + y + x * y + 1/x + 1/(x * y)$ $+1/y + x/y$	A255281, A255282
376	7	[0, 1, 1, 1, 1, 1, 1, 1, 0]	$1 + y + x * y + 1/x + x$ $+1/(x * y) + 1/y$	A247666, A102900
537	7	[1, 0, 1, 0, 1, 1, 1, 1, 1]	$1 + y/x + x * y + x + 1/(x * y)$ $+1/y + x/y$	A255283, A255284
557	7	[1, 0, 1, 1, 0, 1, 1, 1, 1]	$y/x + x * y + 1/x + x$ $+1/(x * y) + 1/y + x/y$	same as Odd-rule 376
575	7	[1, 0, 1, 1, 1, 1, 1, 0, 1]	$1 + y/x + x * y + 1/x + x$ $+1/(x * y) + x/y$	A246039, A246038
377	8	[0, 1, 1, 1, 1, 1, 1, 1, 1]	$1 + y + x * y + 1/x + x$ $+1/(x * y) + 1/y + x/y$	A255275, A255276
577	8	[1, 0, 1, 1, 1, 1, 1, 1, 1]	$1 + y/x + x * y + 1/x + x$ $+1/(x * y) + 1/y + x/y$	A253104, A253105
757	8	[1, 1, 1, 1, 0, 1, 1, 1, 1]	$y/x + y + x * y + 1/x + x$ $+1/(x * y) + 1/y + x/y$	A160239, A246030
777	9	[1, 1, 1, 1, 1, 1, 1, 1, 1]	$(1/x + 1 + x)$ $*(1/y + 1 + y)$	A246035, A139818

Appendix

For each of the 48 combinatorially inequivalent height-one neighborhoods (see Theorem 4 and Tables 1, 2, 3) this Appendix gives the Odd-rule number, the number in [8] of the $b_n(F)$ sequence, and a generating function for that sequence. (In two or three cases, for example Odd-rule 007, the sequence in [8] has an extra initial term compared with the $b_n(F)$ sequence.)

```
Zero cells:
000: (A000004) 1

One cell:
001: (A000012) 1/(1-x)

Two cells:
003: (A000079) 1/(1-2*x)

Three cells:
007: (A001045) (1+2*x)/((1+x)*(1-2*x))
013: (A000244) 1/(1-3*x)

Four cells:
017: (A087206) (1+2*x)/(1-2*x-4*x^2)
033: (A000302) 1/(1-4*x)
035: (A027649) (1-x)/((1-2*x)*(1-3*x))
143: (A255299) (1-x)*(1-x+x^2-x^3-4*x^4+2*x^5-2*x^6)
          /(1-6*x+10*x^2-4*x^3-3*x^4+12*x^5-20*x^6+10*x^7-4*x^8)
146: (A255303) (1-x+2*x^2+2*x^3)/((1-3*x-2*x^2)*(1-2*x+2*x^2))
161: (A255301) (1-x)*(1+x+2*x^2)/(1-4*x+x^2+2*x^3+4*x^4)

Five cells:
037: (A001834) (1+x)/(1-4*x+x^2)
057: (A007483) (1+2*x)/(1-3*x-2*x^2)
117: (A255442) (1+3*x)*(1-x)/((1-3*x)*(1-3*x^2))
147: (A255444) (1-x)*(1+2*x+7*x^4+4*x^5+2*x^6)
          /(1-4*x-x^2+8*x^3+7*x^4-26*x^5+11*x^6+14*x^7+2*x^8-4*x^9)
153: (A255455) (1-x-5*x^2+9*x^3-12*x^4+14*x^5-4*x^6+8*x^7)
          /(1-6*x+6*x^2+20*x^3-51*x^4+56*x^5-46*x^6+20*x^7-8*x^8)
156: (A255453) (1-x+2*x^2-4*x^3)/((1-x)*(1-5*x+6*x^2-4*x^3))
163: (A255457) (1-x)*(1+x-x^2+x^3)/(1-5*x+24*x^3-15*x^4-17*x^5)
165: (A255447) (1-x)*(1+x)*(1+x-x^2)/((1-x-x^2)*(1-3*x-5*x^2+11*x^3))
166: (A255451) (1+x)/(1-4*x-x^2+4*x^3+8*x^4)
171: (A253067) (1+2*x)*(1+2*x+3*x^2+4*x^3)/(1-x-5*x^2-13*x^3-6*x^4-8*x^5)
255: (A255459) (1-x+6*x^2)/((1-x)*(1-2*x)*(1-3*x))
345: (A255449) (1-x)*(1-x^2-2*x^3-6*x^4)/(1-6*x+10*x^2-8*x^3+15*x^4-10*x^5-10*x^6)

Six cells:
077: (A246036) (1+4*x)/((1+2*x)*(1-4*x))
137: (A255465) (1+3*x)/((1+x)*(1-4*x))
```

```
157: (A255469) (1+x-9*x^2+15*x^3+2*x^4-34*x^5+20*x^6-16*x^7-8*x^8)
           /(1-5*x+x^2+25*x^3-44*x^4+2*x^5+56*x^6-40*x^7+24*x^8+16*x^9)
167: (A255467) (1+2*x)*(1-x)/((1-4*x)*(1-2*x)*(1+x))
173: (A255476) (1+2*x)*(1+x-2*x^2-x^3-5*x^4-7*x^5+2*x^6-7*x^7+6*x^8)
           /(1-3*x-6*x^2+9*x^3+9*x^4+7*x^5-2*x^6-15*x^7+4*x^8+8*x^9+8*x^10)
175: (A253070) (1+2*x)*(1+x-x^2+x^3+2*x^5)
           /(1-3*x-3*x^2+x^3+6*x^4-10*x^5+8*x^6-8*x^7)
176: (A255471) (1+3*x)/((1-x)*(1+2*x)*(1-4*x))
257: (A255474) (1-8*x^2-16*x^3)/((1-4*x)*(1-2*x-4*x^2))
275: (A253068) (1+3*x+4*x^2)/((1-x)*(1+2*x)*(1-4*x))
347: (A253101) (1-3*x^2+4*x^3)/((1-2*x)*(1-4*x+x^2))
356: (A164908) (1+2*x)/(1-4*x)
365: (A255463) (1-x)/((1-3*x)*(1-4*x))
517: (A255461) (1-x)*(1+2*x-3*x^2-6*x^3-2*x^4+4*x^5)
           /((1+x)*(1-2*x)*(1-4*x+x^2+2*x^4-4*x^5))
555: (A255296) (1+2*x)*(1-x)/((1-2*x)*(1-3*x-2*x^2))

Seven cells:
177: (A255278) (1+4*x-3*x^3+6*x^4-6*x^5-12*x^6-12*x^7)
           /((1+x)*(1-4*x-2*x^2+9*x^3+2*x^4+2*x^5-8*x^6+12*x^7))
277: (A255280) (1+2*x-7*x^2+12*x^3-16*x^5-16*x^6)
           /(1-5*x+3*x^2+9*x^3-16*x^4+16*x^6+16*x^7)
357: (A253072) (1+x-16*x^2+28*x^3-8*x^4)/(1-6*x+5*x^2+24*x^3-44*x^4+8*x^5)
367: (A255282) (1+2*x-4*x^2-7*x^3+5*x^4-2*x^5+9*x^7-2*x^8+6*x^9)
           /(1-5*x+21*x^3-18*x^4-3*x^5+24*x^6-31*x^7+11*x^8-22*x^9-10*x^10)
376: (A102900) (1+4*x)/((1+x)*(1-4*x))
537: (A255284) (1+4*x)*(1-x)/((1-4*x)*(1-7*x^2))
575: (A246038) (1+2*x)*(1+2*x+4*x^2)/(1-3*x-8*x^3-8*x^4)

Eight cells:
377: (A255276) (1-20*x^2+56*x^3-49*x^4-36*x^5+128*x^6-128*x^7)
           /(1-8*x+16*x^2+24*x^3-145*x^4+236*x^5-164*x^6+24*x^7+32*x^8)
577: (A253105) (1+4*x+6*x^3+x^4+8*x^5-4*x^6)
           /(1-4*x-6*x^3+29*x^4-12*x^5-8*x^6+8*x^7)
757: (A246030) (1+6*x)/((1+2*x)*(1-4*x))

Nine cells:
777: (A139818) (1+6*x-8*x^2)/((1-x)*(1+2*x)*(1-4*x))
```

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