On a Conjecture of Melkamu Zeleke

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In his fascinating talk[Z], Melkamu Zeleke made the following conjecture.

Conjecture: Let $C(z)$ be the generating function of the Catalan numbers, i.e. the formal power series satisfying

$$C(z) = 1 + zC(z)^2 \quad ,$$

then for all positive integers m, the formal power series

$$F_m(z) := \frac{zC^2 - (zC^2)^m}{1 - (zC^2)^{m+1}} \quad ,$$

are all **rational functions** of z.

In this short note we will give an explicit description of $F_m(z)$ as a rational function, that, once conjectured, can be **rigorously** proved by **only** checking finitely many special cases. It turns out that checking it for $2 \le m \le 10$ suffices! (why?). Just to play it safe I checked the proposition below for $2 \le m \le 100$.

Proposition Let $N_m(z)$ be the coefficient of X^m in the Maclaurin expansion of

$$\frac{-z - zX + z^2X^2}{1 + (1 - 2\,z)\,X^2 + z^2X^4} \quad ,$$

and Let $D_m(z)$ be the coefficient of X^m in the Maclaurin expansion of

$$\frac{-1 + z + (2\,z - 1)\,X - z^2X^2 - z^2X^3}{1 + (1 - 2\,z)\,X^2 + z^2X^4} \quad ,$$

then

$$F_m(z) = \frac{N_{m-2}(z)}{D_{m-2}(z)} \quad . \quad \square$$

Reference

[Z] Melkamu Zeleke, *On Subsets of Ordered Trees Enumerated by a Subsequence of Fibonacci Numbers* , talk delivered at the Rutgers Experimental Mathematics Seminar, April 18, 2013.

(Based on joint work with Mahendra Jani)

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