

**There are  $\frac{1}{30}(r+1)(r+2)(2r+3)(r^2+3r+5)$  Ways For the Four Teams of a World Cup Group  
to Each Have  $r$  Goals For and  $r$  Goals Against  
[Thanks to the Soccer Analog of Prop. 4.6.19 of Richard Stanley's (Classic!) EC1]**

*By Shalosh B. EKHAD and Doron ZEILBERGER*

*Dedicated to Richard Peter Stanley who just turned "number of ways for a simple  
1D Drunkard to return home after 8 steps"-years-old*

**Proof of the Statement in the Title:** Calling this quantity  $S_4(r)$ , and using the case  $n = 4$  of the Soccer analog of Prop. 4.6.19 of [EC1] (see Comment 1 below), we see that

- $S_4(r)$  is a polynomial of degree 5 .
- $S_4(-1) = 0$  ,  $S_4(-2) = 0$  .
- $S_4(-3-r) = -S_4(r)$  .

Hence, it suffices to check the statement at the **two** values  $r = 0$  and  $r = 1$ . Obviously  $S_4(0) = 1$  (there is just one all-zero  $4 \times 4$  matrix), and almost obviously,  $S_4(1) = 9$  (the number of derangements of length 4 equals 9). QED!

**Comments**

1. The same method of proof that guru Richard Stanley used to prove Prop. 4.6.19 of [EC1] (first edition; it is Prop. 4.6.2 in the second edition.) [the first edition is missing  $(-1)^{n-1}$  in front of  $H_n(r)$ ] yields:

**The Soccer analog of Prop. 4.6.19 of EC1**

Let  $S_n(r)$  be the number of ways  $n$  Soccer teams, playing a round-robin tournament, each scored a total of  $r$  "Goals For" (GF), and a total of  $r$  "Goals Against" (GA), (in other words the number of  $n \times n$  magic squares with line-sums  $r$  and whose diagonal entries are all 0 (no team plays against itself!)).

For each fixed integer  $n \geq 3$ , the function  $S_n(r)$  is a polynomial in  $r$  of degree  $n^2 - 3n + 1$ . Since it is a polynomial, it can be evaluated at *any* (not necessarily positive) integer, and we have

$$S_n(-1) = S_n(-2) = \dots = S_n(-n+2) = 0 \quad ,$$

$$S_n(-(n-1)-r) = -S_n(r) \quad .$$

2. We got the idea for this tribute to our beloved guru when we attended the Stanley@70 conference, held at MIT, June 23-27, 2014, at the same time as the preliminary Group stage of the World Cup 2014.

**3.** While the proof of the statement of the title did not require any computer, the analogous result for  $n = 5$  is already beyond the scope of mere humans (or the human would have to be very stupid to spend time on it). We have

$$S_5(r) = \frac{1}{241920} (r+1)(r+2)(r+3) \cdot$$

$$(43r^8 + 688r^7 + 4934r^6 + 20680r^5 + 55907r^4 + 101272r^3 + 123436r^2 + 96240r + 40320) \quad .$$

Of course  $S_3(r) = r + 1$  (why?).

To see  $S_6(r)$ , please go to: <http://www.math.rutgers.edu/~zeilberg/tokhniot/oGOALS1> .

One can get this, and (potentially!) infinitely more results, using the Maple package **GOALS** available directly from: <http://www.math.rutgers.edu/~zeilberg/tokhniot/GOALS> , or via the front of this article: <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/worldcup.html> .

In particular, procedure `MagicPolAG(n,r,Sr,Sc,B,A)` can find polynomial expressions for the number of  $n \times n$  matrices with non-negative integer entries, whose row-sums are

$$(r + Sr[1], \dots, r + Sr[n]) \quad ,$$

and whose column sums are

$$(r + Sc[1], \dots, r + Sc[n]) \quad ,$$

for *any* fixed numerical vectors  $Sr$ ,  $Sc$ , of length  $n$  (of course they have to add-up to the same number), and any assignment  $A$  where some entries are fixed beforehand, where  $B$  denotes “wild-card”. See the on-line help there. See the above-mentioned front of this article for numerous sample input and output files.

**4.** The sequence for  $S_4(r)$  is already in Neil Sloane’s OEIS, (see [OEIS1]), but for **different** reasons! Can you find a bijection? On the other hand, the sequence for  $S_5(r)$  is not (yet!) there:

$$1, 44, 870, 9480, 68290, 365936, 1573374, 5709120, 18107760, 51488800, 133748186, 321979164, \dots \quad ,$$

but we are sure that very soon it will!

**5.** Another, even more useful, Maple package accompanying this article is `WorldCup`, one of whose many procedures is ‘`Ptor`’, that finds all the possible scenarios that lead to a given *score board*, consisting of the “Goals For”, “Goals Against”, and “PTS”, vectors. (Recall that a win yields 3 points, a draw, 1 point, and a loss, 0 points.)

The number of possible scenarios for groups A-H were as follows

**2014:** 8, 32, 7, 3, 13, 3, 12, 3 .

**2010:** 2, 3, 2, 2, 3, 3, 1, 2 (in particular, the score-board for Group  $G$  *uniquely* determined the individual game scores).

**2006:** 11, 1, 5, 3, 2, 6, 1, 8 (in particular, the score-boards for Groups  $B$  and  $G$  *uniquely* determined the individual game scores).

**2002:** 1, 12, 5, 5, 1, 2, 3, 1 (in particular, the score-boards for Groups  $A, E, H$  *uniquely* determined the individual game scores).

**1998:** 9, 6, 4, 3, 3, 3, 1, 1 (in particular, the score-boards for Groups  $G, H$  *uniquely* determined the individual game scores).

**6.** The Israeli daily *Yedioth Ahronoth* published, at the start of the 2014 World Cup, a “soccer puzzle” where the above-mentioned vectors, GF, GA, and PTS, are given, and the solver has to reconstruct the scores of the individual matches. The above-mentioned Maple package `WorldCup` has a procedure, `Khida`, that makes up such puzzles, and another procedure, `Sefer1`, that creates challenging puzzle books. For some samples, see

<http://www.math.rutgers.edu/~zeilberg/tokhniot/oWorldCupi> , for  $1 \leq i \leq 4$ . Enjoy!

**7.** Happy birthday Richard, and keep up the good work! (and continue to practice what Phil Hanlon calls the “the three H’s”).

## References

[OEIS1] Neil J.A. Sloane, *Sequence A061927*, <http://oeis.org/A061927>.

[EC1] Richard P. Stanley, “*Enumerative Combinatorics, volume I*”, first edition: Wadsworth & Brooks/Cole, 1986; second edition: Cambridge University Press, 2011.

Available on-line (viewed July 7, 2014): <http://math.mit.edu/~rstan/ec/ec1.pdf> .

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