The Experimental Mathematics of Voting

By Shalosh B. EKHAD and Doron ZEILBERGER

By using the Maple package http://www.math.rutgers.edu/~zeilberg/tokhniot/LinDiophantus one derives the following theorem (in less than 20 seconds, see `http://www.math.rutgers.edu/~zeilberg/tokhniot/oLinDiophantus2`), the following theorem.

Theorem 1 The weighted generating function of The Condorcet scenario $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ (with three candidates $1, 2, 3$ and according to the weight

$$x123^{NumberOf123} \cdots x321^{NumberOf321} t^{N} umberOfVotes$$

is

$$\frac{\left(1 + t^3 x123\ x231\ x312\right) t^3 x312\ x123\ x231}{(1 - t^2 x123\ x321)(1 - t^2 x213\ x312)(1 - t^2 x312\ x123)(1 - t^2 x132\ x231)(1 - t^2 x123\ x231)(1 - t^2 x231\ x312)} \quad .$$

By applying the 'umbra'

$$x123^{a123} \cdots x321^{a321} \rightarrow (a123 + \ldots + a321)!/((a123! \cdots a321!) \cdot 6^{a123 + \ldots + a321})$$

to the above one generating functions, one gets a differential equation (via WZ-theory) that translates into a recurrence that translates (for the odd part) to the following theorem.

Theorem 2: Suppose three candidates, 1,2, and 3, are running for office, and there are $2n - 1$ voters, each of them choosing one of the 3!=6 possible ranking with the same probability (1/6). The votes are counted and it turns out that in the vote of

1 vs. 2, 1 won

2 vs. 3, 2 won

1 vs. 3, 3 won

In other words, there is a cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.

Let a(n) be the probability of that happening times 6^{2n-1} (i.e. the numerator)

The integer sequence, $a(n)$, satisfies the following linear recurrence equation with polynomial coefficients

$$-1296 \frac{n(2n+3)(2n+1)a(n)}{(n+1)(n+2)^2} + 36 \frac{(2n+3)\left(22n^2+33n+12\right)a(n+1)}{(n+1)(n+2)^2}$$
$$-4 \frac{\left(19n^2+57n+45\right)a(n+2)}{(n+2)^2} + a(n+3) = 0 \quad .$$

The asymptotic expression for $a(n)$ is $0.043869914022955 - 0.021101164\, n^{-1} + O(n^{-2})$.

By symmetry, this is also the probability of the scenario $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$. So doubling the asympotic expression for the Condorcet scenario is

$$0.087739828045910 - 0.042202328\, n^{-1} \quad .$$

Corollary: The probability for a strict Condorcet scenario with three candidates and $2n-1$ voters tends as $n \rightarrow \infty$ to a constant, that is approximately equal to $0.087739828045910\ldots$, and that should be named the Condorcet constant.

Comment This number was first derived by cosnsiderable human effort by W.V. Gehrlein and P.C. Fishburn, *The probability of paradox of voting: a computable solution*, J. of Economic Theory **13** (1976), 14-25 [bottom of page 17, left column, where the complementary probability, 0.91226 (only to five digits) is given]

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA;
DoronZeil at gmail dot com ; `http://www.math.rutgers.edu/~zeilberg/` .

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