A 2-COLORING OF $[1,N]$ CAN HAVE $(1/22)N^2+O(N)$ MONOCHROMATIC SCHUR TRIPLES, BUT NOT LESS!

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Abstract: We prove that the minimum number (asymptotically) of monochromatic Schur triples that a 2-coloring of $[1,n]$ can have is $\frac{n^2}{22}+O(n)$. This revised version fills in a minor and subtle gap discovered by M. Primak. (The revision also corrects (at no extra cost) a discrepancy between the solution in the paper and the solution obtained by Maple. In the paper $H_{1/2}$ should be H_0 and H_1 should be $H_{1/2}$ for the solutions to agree.)

Tianjin, June 29, 1996: In a fascinating invited talk at the SOCA 96 combinatorics conference organized by Bill Chen, Ron Graham proposed (see also [GRR], p. 390):

Problem ($100): Find (asymptotically) the least number of monochromatic Schur triples $\{i,j,i+j\}$ that may occur in a 2-coloring of the integers $1,2,\ldots,n$.

By renaming the two colors 0 and 1, the above is equivalent to the following

Discrete Calculus Problem: Find the minimal value of

$$F(x_1,\ldots,x_n) := \sum_{\substack{1\le i<j\le n\\ i+j\le n}} [\ x_ix_jx_{i+j}+(1-x_i)(1-x_j)(1-x_{i+j})\ ]\ ,$$

over the n-dimensional (discrete) unit cube $\{(x_1,\ldots,x_n)|x_i=0,1\}$. We will determine *all* local minima (with respect to the Hamming metric), then determine the global minimum.

Partial Derivatives: For any function $f(x_1,\ldots,x_n)$ on $\{0,1\}^n$ define the discrete *partial derivatives* $\partial_r f$ by $\partial_r f(x_1,\ldots,x_r,\ldots,x_n) := f(x_1,\ldots,x_r,\ldots,x_n) - f(x_1,\ldots,1-x_r,\ldots,x_n)$.

If $(z_1,\ldots,z_n)$ is a local minimum of F, then we have the n inequalities:

$$\partial_r F(z_1,\ldots,z_n) \le 0 \quad , \quad 1\le r\le n.$$

A purely routine calculation (applicable Maple routines: `diff1, dif`) shows that (below $\chi(S)$ is 1(0) if S is true(false))

$$\partial_r F(x_1,\ldots,x_n) =$$

$$(2x_r-1)\left\{\sum_{i=1}^{n}x_i+\sum_{i=1}^{n-r}x_i-(n-\left\lfloor\frac{r}{2}\right\rfloor)-\chi(r>\frac{n}{2})-(2x_r-1)+x_r\chi(r>\frac{n}{2})+1-(x_{\frac{r}{2}}+x_{2r})\chi(r\le\frac{n}{2})\right\}.$$

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Since we are only interested in the *asymptotic* behavior, we can modify F by any amount that is $O(n)$. In particular, we can replace $F(x_1,\ldots,x_n)$ by

$$G(x_1,\ldots,x_n) = F(x_1,\ldots,x_n) + \sum_{i=1}^{n/2} x_i(x_{2i}-1) - \frac{1}{2}\sum_{i=1}^{n} x_i.$$

Noting that $(2x_r-1)^2 \equiv 1$ and $(2x_r-1)x_r \equiv x_r$ on $\{0,1\}^n$, we see that for $1 \le r \le n$,

$$\partial_r G(x_1,\ldots,x_n) = (2x_r-1)\left\{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n-r} x_i - (n - \left\lfloor \frac{r}{2} \right\rfloor) - \frac{1}{2}\chi(r \le n/2)\right\} - \frac{1}{2}\chi(r \le n/2) - 1/2.$$

Let $k = \sum_{i=1}^{n} x_i$. Since at a local minimum $(z_1,\ldots,z_n)$ we have $\partial_r G(z_1,\ldots,z_n) \le 0$, it follows that any local minimum $(z_1,\ldots,z_n)$ satisfies the

Ping-Pong Recurrence: Choose $a, b \in \{0,1\}$ arbitrarily each time $\widehat{H}$ or $\widetilde{H}$ is used, where $\widehat{H}$ and $\widetilde{H}$ are the following functions:

$$\widehat{H}(y) := \begin{cases} 0, & \text{if } y > 1/2; \\ 1, & \text{if } y < 0; \\ a, & \text{if } 0 \le y \le 1/2. \end{cases}$$

$$\widetilde{H}(y) := \begin{cases} 0, & \text{if } y > 1; \\ 1, & \text{if } y < -1; \\ b, & \text{if } -1 \le y \le 1. \end{cases}$$

Then we must have, for $r = n, n-1, \ldots, n - \lfloor n/2 \rfloor + 1$,

$$z_r = \widehat{H}\left(k - n + \left\lfloor \frac{r}{2} \right\rfloor + \sum_{j=1}^{n-r} z_j\right), \qquad (Right\,Volley)$$

$$z_{n-r+1} = \widetilde{H}\left(2k - n - 1/2 + \left\lfloor \frac{n-r+1}{2} \right\rfloor - \sum_{j=r}^{n} z_j\right), \qquad (Left\,Volley)$$

and if n is odd then $z_{(n+1)/2} = \widehat{H}(k - n + \lfloor \frac{n+1}{4} \rfloor + \sum_{j=1}^{(n-1)/2} z_j)$.

These equations determine a solution (depending upon the choices of the a's and b's made along the way), z (if it exists), in the order $z_n, z_1, z_{n-1}, z_2, \ldots$. When we solve the Ping-Pong recurrence we forget the fact that $\sum_{i=1}^{n} z_i = k$. Most of the time a solution will not satisfy this last condition, but when it does, we have a genuine local minimum. Note that *any* local minimum must show up in this way.

Solutions of the Ping-Pong Recurrence: By playing around with the Maple routine `ptor2` in our Maple package `RON` (available from either author's website), we were able to find the following solutions, *for n sufficiently large*, to the Ping-Pong recurrence. As usual, for any word (or letter) W, W^m means 'W repeated m times'.

Let $w = 2k - n$, $k \neq n/2$ (this case must be dealt with separately). By symmetry we may assume that $k \geq n/2$. Then $0 < w \leq n$. If $w \geq n/2$ then the only solution is 0^n. If $w < n/2$, then let s be the unique integer $0 \leq s < \infty$, that satisfies $n/(12s+14) \leq w < n/(12s+2)$.

Case I: If $n/8 \leq w < n/2$ then the solutions are:

$$0^{\lfloor \frac{n}{2} \rfloor} 1^{n - \lfloor \frac{n}{2} \rfloor - w - c_1} 0^{w + c_1}$$

where $c_1 \in \{-1, 0, 1\}$.

Case II: If $n/(12s+8) \leq w < n/(12s+2)$ then the solutions are

$$\begin{cases} 0^{4w+c_1} 1^{\lfloor n/2 \rfloor - 4w - c_1} 0^{n - \lfloor n/2 \rfloor - 7w - (c_2+c_3+c_4)} 1^{6w+c_3} 0^{w+c_4} & \text{for } s = 1; \\ 0^{4w+c_4} (1^{6w+c_5^{s_i}} 0^{6w+c_6^{s_i}})^{s/2} Q (0^{6w+c_7^{s_i}} 1^{6w+c_8^{s_i}})^{s/2} 0^{w+c_9} & \text{for } s > 1. \end{cases}$$

where the c_j's and $c_j^{s_i}$'s are bounded constants (independent of n) and Q can be an (almost) arbitrary mix of r zeroes and ones (where r is the unique integer such that the length of this interval is n). Further, the number of ones in Q is at most $12w$. Notation: (1) the $c_j^{s_i}$'s can be different constants with i ranging from 1 to $s/2$; (2) if s is odd $(ab)^{s/2}$ is $(ab)^{(s-1)/2}a$.

Case III: If $n/(12s+14) \leq w < n/(12s+8)$ then the solutions are

$$\begin{cases} 0^{4w+d_1} 1^{n-5w-(d_1+d_2)} 0^{w+d_2} & \text{for } s = 0; \\ 0^{4w+d_3} (1^{6w+d_4^{s_i}} 0^{6w+d_5^{s_i}})^{s/2} Q (0^{6w+d_6^{s_i}} 1^{6w+d_7^{s_i}})^{s/2} 0^{w+d_8} & \text{for } s > 0. \end{cases}$$

where the d_j's and $d_j^{s_i}$'sare bounded constants (independent of n) and Q can be an (almost) arbitrary mix of r zeroes and ones, with the number of ones in Q at most $6w$.

Case IV: if $w = 0$ (i.e. $s = \infty$), the solutions are:

$$0^{g_1} (1^{g_2^{n_i}} 0^{g_3^{n_i}})^{n/(2G_1)} Q (0^{g_4^{n_i}} 1^{g_5^{n_i}})^{n/(2G_2)}$$

where $g_1 \in \{0, 1, 2\}$, the other g_i's and $g_i^{n_i}$'s are bounded between 3 and 11, Q is an (almost) arbitrary mix of r zeroes and ones with the number of ones bounded between 0 and 22, $G_1 = \sum_i (g_2^{n_i} + g_3^{n_i})$, and $G_2 = \sum_i (g_4^{n_i} + g_5^{n_i})$.

Proof: Routine verification!

Now it is time to impose the extra condition that $\sum_{i=1}^{n} z_i = k$ $(= (w+n)/2)$. With Cases I and II a routine calculation yields a contradiction of the applicable range of w when n is sufficiently large. For Case III, a routine calculation yields a local minimum of $w = n/11$ if $s = 0$. If $s > 0$ argue as follows. Let t be the number of 1's in Q. Recall that r is the total number of 0's and 1's in Q. Let $w_c(s) = n/(12s+c)$ where we must have $8 \leq c \leq 14$. Since we need $\sum_{i=1}^{n} z_i = k$ $(= (w+n)/2)$, we see that $6w_c(s)s + t = n(12s+c+1)/(24s+2c)$ gives $t = (c+1)w_c(s)/2$. Further, since the number of 1's in Q is bounded by $6w_c(s)$, we find that we must have $c \leq 11$. We also must have $r = n - w_c(s)(12s+5)$, by the definition of r. Using the simple inequality $r \geq t$, we have

$n - w_c(s)(12s+5) \ge (c+1)w_c(s)/2$. From this deduce that $c \ge 11$. Hence we must have $c = 11$ at a local minimum. Thus the local minimums for Case III, $s > 0$, are $w_s = n/(12s+11)$. Case IV gives infinitely many local minimums. Hence

The Local Minima Are Asymptotically Equivalent (mod $O(n)$) to:

$$\begin{cases} Z_s := 0^{4w_s}(1^{6w_s}0^{6w_s})^{\frac{s}{2}}1^{6w_s}(0^{6w_s}1^{6w_s})^{\frac{s}{2}}0^{w_s} & \text{for } 0 \le s < \infty \text{ (where } w_s := \frac{n}{12s+11}), \\ Z_\infty^t = (0^t1^t)^{n/(2t)} & \text{for } 3 \le t \le 11 \end{cases}$$

A routine calculation [R] shows that for $0 \le s < \infty$

$$F(Z_s) = \frac{12s+8}{16(12s+11)}n^2 + O(n),$$

which is strictly increasing in s. An easy calculation shows $F(Z_\infty^t) = (1/16)n^2 + O(n)$ for any natural number t.

...And The Winner Is: $Z_0 = 0^{4n/11}1^{6n/11}0^{n/11}$ setting the world-record of $(1/22)n^2 + O(n)$.

An Extension Here we note that our result implies a good upper bound for the general r-coloring of the first n integers; if we r-color the integers (with colors $C_1 \ldots C_r$) from 1 to n then the minimum number of monochromatic Schur triples is bounded above by

$$\frac{n^2}{2^{2r-3}11} + O(n).$$

This comes from the following coloring:

$$\begin{cases} Color(i) = C_j & \text{if } \frac{n}{2^j} < i \le \frac{n}{2^{j-1}} \quad \text{for } 1 \le j \le r-2, \\ Color(i) = C_{r-1} & \text{if } 1 \le i \le \frac{4n}{2^{r-2}11} \text{ or } \frac{10n}{2^{r-2}11} < i \le \frac{n}{2^{r-2}}, \\ Color(i) = C_r & \text{if } \frac{4n}{2^{r-2}11} < i \le \frac{10n}{2^{r-2}11}. \end{cases}$$

Note: Tomasz Schoen[S], a student of Tomasz Luczak, has independently solved this problem.

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