

# A CONDENSED CONDENSATION PROOF OF A DETERMINANT EVALUATION CONJECTURED BY Greg KUPERBERG AND Jim PROPP

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Greg Kuperberg and Jim Propp [P] have conjectured the following determinant identity:

$$\det \left[ \binom{i+j}{i} \binom{2n-i-j}{n-i} \right]_{0 \leq i, j \leq n} = \frac{(2n+1)!^{n+1}}{(2n+1)!!} ,$$

where  $a!! := 0! \cdot 1! \cdot 2! \cdots a!$ , and  $a! := 1 \cdot 2 \cdots a$ .

This is the special case ( $m = n, a = b = 0$ ) of

$$\det \left[ \binom{i+j+a+b}{i+a} \binom{2n-i-j-a-b}{n-i-a} \right]_{0 \leq i, j \leq m} = \frac{(a+b)!(2n+1)!^{m+1}(2n-m)!!m!!(m+a+b)!!(2n-m-a-b)!!a!!b!!(n-m-a-1)!!(n-m-b-1)!!}{a!b!(2n+1)!!(n-a)!!(n-b)!!(m+a)!!(m+b)!!(a+b)!!(2n-2m-a-b-1)!!} ,$$

*(Rabbit)*

which follows immediately from Dodgson's[D] rule for evaluating determinants: (For any  $n \times n$  matrix  $A$ , let  $A_r(k, l)$  be the  $r \times r$  connected submatrix whose upper leftmost corner is the entry  $a_{k,l}$ .)

$$\det A = \frac{\det A_{n-1}(1, 1) \det A_{n-1}(2, 2) - \det A_{n-1}(1, 2) \det A_{n-1}(2, 1)}{\det A_{n-2}(2, 2)} . \quad \text{(*Lewis*)}$$

Indeed, let the left and right sides of *(Rabbit)* be  $L_m(a, b)$  and  $R_m(a, b)$  respectively. Dodgson's rule immediately implies that the recurrence:

$$X_m(a, b) = \frac{X_{m-1}(a, b)X_{m-1}(a+1, b+1) - X_{m-1}(a+1, b)X_{m-1}(a, b+1)}{X_{m-2}(a+1, b+1)} ,$$

holds with  $X = L$ . Since  $L_m(a, b) = R_m(a, b)$  for  $m = 0, 1$  (check!), and the recurrence also holds with  $X = R$  (check!<sup>2</sup>), it follows by induction that  $L_m(a, b) = R_m(a, b)$  for *all*  $m$ .  $\square$ .

The present proof is in the spirit of [Z1]. Another proof can be found in [A1]. The same method yields a q-analog of *(Rabbit)*, that can be found in [A2]. A beautiful combinatorial proof of *(Lewis)* can be found in [Z2]. An alternative proof of *(Rabbit)* is given in [K].

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<sup>2</sup> Divide both sides by the left, then use  $r!/(r-1)!!=r!$  whenever possible, and then  $r!/(r-1)!!=r$  whenever possible, reducing it to a completely routine polynomial identity. The small Maple package `rabbit` obtainable from our Home Pages, performs these steps mechanically.

## References

- [A1] T. Amdeberhan, *A WZ proof of a determinant evaluation conjectured by Kuperberg and Propp*, exclusively published in Amdeberhan's Home Page <http://www.math.temple.edu/~tewodros>.
- [A2] T. Amdeberhan, *A q-generalization of a determinant evaluation conjectured by Kuperberg and Propp*, exclusively published in Amdeberhan's Home Page <http://www.math.temple.edu/~tewodros>.
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- [K] C. Krattenthaler, *E-mail message to T. Amdeberhan*, dated 9 Aug. 1996, 19 : 22 : 27 + 0100 (MET). By kind permission of Krattenthaler, it can be gleaned at the first author's Home Page <http://www.math.temple.edu/~tewodros>.
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- [Z1] D. Zeilberger, *Reverend Charles to the aid of Major Percy and Fields Medalist Enrico*, Amer. Math. Monthly **103**(1996), 501-502.
- [Z2] D. Zeilberger, *Dodgson's determinant-evaluation rule proved by Two-Timing Men and Women*, to appear in the Elec. J. of Combinatorics [Wilf Festschrift volume]. It can be downloaded from Zeilberger's Home Page: <http://www.math.temple.edu/~zeilberg>.

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