

# Q-APÉRY IRRATIONALITY PROOFS BY Q-WZ PAIRS

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ABSTRACT. Using WZ pairs, Apéry-style proofs of the irrationality of the q-analogues of the Harmonic series and  $\text{Ln}(2)$  are given.

## 0. Introduction:

Let us define the following q-analogues of the Harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  and  $\text{Ln}(2)$ , respectively by:

$$(0.1) \quad h_q(1) := \sum_{k=1}^{\infty} \frac{1}{q^k - 1} \quad (\text{for } |q| > 1),$$

and

$$(0.2) \quad \text{Ln}_q(2) := \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} \quad (\text{for } |q| \neq 0, 1).$$

In 1948, Paul Erdős [E1] proved the irrationality of  $h_2(1)$ . Recently, Peter Borwein used Pade approximation techniques [B1] and some complex analysis methods [B2] to prove the irrationality of both  $h_q(1)$  and  $\text{Ln}_q(2)$ . Here we present a proof in the spirit of Apéry's magnificent work [A], which was delightfully accounted by Alf van der Poorten [P]. This method of proof gives favorable irrationality measure for  $\text{Ln}_q(2)$  compared with those implied in [B1], [B2]. Further discussion of irrationality results for certain series is to be found in Erdős [E2].

We will assume familiarity with ref. [Z]. In particular, we use

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \text{ where } (q)_n = (1-q) \cdots (1-q^n).$$

N and K are *forward shift operators* on n and k, respectively.

$$\Delta_n = N - 1, \Delta_k = K - 1.$$

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### 1. A scheme for proving the irrationality of the $q$ -harmonic series $h_q(1)$ :

The claims made in subsections 1.1-1.5 below were found using the Maple Package **qEKHAD** accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

1.1. The  $q$ -WZ 1-form  $\omega$  is:

$$\omega = \frac{-1}{\binom{n+k+1}{k}_q (q)_{n+1}} \left\{ \delta k + \frac{q^{n+1}}{(q^{n+1} - 1)} \delta n \right\}.$$

1.2. The choice of the potential  $c(n, k)$  is:

$$c(n, k) = \sum_{m=1}^n \frac{q^m}{(1 - q^m)(q)_m} + \sum_{m=1}^k \frac{1}{(q^m - 1)} \frac{1}{\binom{n+m}{m}_q (q)_n}.$$

1.3. The choice of the mollifier  $b(n, k)$  is:

$$b(n, k) = (-1)^k q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q.$$

1.4. We define two sequences:

$$a(n) = \sum_{k=0}^n c(n, k) b(n, k), \quad \text{and} \quad b(n) = \sum_{k=0}^n b(n, k).$$

1.5. Introduce  $L = y_2(n)N^2 + y_1(n)N + y_0(n)$  and  $B(n, k) = P_q^1(n, k)b(n+1, k)$ , where

$$A(n, k) = c(n, k)B(n, k) + \frac{(-1)^k q^{2n+3}}{q^{n+1} - 1} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}}}{(q)_{n+2}} P_q^2(n, k) \quad \text{and}$$

$$P_q^1(n, k) = -q\alpha_n^2 \beta_k^{-1} (q^2 \alpha_n + 2q) + q\alpha_n^2 (q^2 \alpha_n^3 + 2q(q+1)\alpha_n^2 + 3q\alpha_n - (q+1) - (\alpha_n + 2)\beta_k)$$

$$P_q^2(n, k) = q^2 \alpha_n^2 + q\alpha_n - 2 + \beta_k (q^2 \alpha_n^5 + q(2q+1)\alpha_n^4 - 2\alpha_n^3 - \alpha_n^3 \beta_k - (2 - q^{-1})\alpha_n^2 \beta_k - (3q+5)\alpha_n^2 + 2q^{-1}\alpha_n \beta_k + (q-1+2q^{-1})\alpha_n + (1+3q^{-1})),$$

$$y_0(n) = q(\alpha_n - 1)(q\alpha_n + 2), \quad y_2(n) = (q\alpha_n - 1)(\alpha_n + 2), \quad \alpha_n = q^{n+1}, \quad \beta_k = q^{k+1} \quad \text{and}$$

$$y_1(n) = q^3 \alpha_n^5 + 2q^2(q+1)\alpha_n^4 + q^2 \alpha_n^3 - 4q(q+1)\alpha_n^2 + (q^2 - 4q + 1)\alpha_n + 2(q+1).$$

Then

$$(*) \quad L(b(n, k)) = B(n, k) - B(n, k-1) \quad \text{and} \quad L(b(n, k)c(n, k)) = A(n, k) - A(n, k-1).$$

Now, summing over  $k$  in  $(*)$  shows that both sequences  $a(n)$  and  $b(n)$  are solutions of  $Lu(n) = 0$ .

1.6. Set  $b_n = b(n)$  and  $a_n = a(n)$ . Now since  $b_{n+1} > b_n$  and  $Lb_n = 0$ , i.e.  $y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0$ , then asymptotically we have that

$$\frac{b_{n+2}}{b_{n+1}} = O\left(\frac{y_1(n)}{y_2(n)}\right) = O(q^{3n+3}).$$

Hence,

$$(1.6.1) \quad b_n = O\left(q^{\frac{3n^2}{2}}\right).$$

On the other hand,  $La_n = 0$  and  $Lb_n = 0$  lead to the system of recurrence equations,

$$(1.6.2) \quad y_2(n)a_{n+2} + y_1(n)a_{n+1} + y_0(n)a_n = 0, \quad y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0.$$

Multiplying the first and the second equations in (1.6.2), respectively by  $b_{n+2}$  and  $a_{n+2}$ , and subtracting we obtain

$$y_1(n)(a_{n+1}b_{n+2} - b_{n+1}a_{n+2}) = y_0(n)(a_{n+2}b_n - b_{n+2}a_n).$$

Rewriting this

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}} = \frac{y_0(n)}{y_1(n)} \frac{b_n}{b_{n+1}} \left( \frac{a_{n+2}}{b_{n+2}} - \frac{a_n}{b_n} \right).$$

Then we have the estimate

$$\left| \frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}} \right| \leq \left| \frac{y_0(n)}{y_1(n)} \frac{b_n}{b_{n+1}} \left( \frac{a_{n+2}}{b_{n+2}} - \frac{a_{n+1}}{b_{n+1}} \right) \right| + \left| \frac{y_0(n)}{y_1(n)} \frac{b_n}{b_{n+1}} \left( \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} \right) \right|.$$

This yields in

$$(1.6.3) \quad \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = O(b_n^{-2}).$$

Therefore,

$$(1.6.4) \quad h_q(1) - \frac{a_n}{b_n} = O(b_n^{-2}).$$

In particular, the sequence of rational numbers  $\frac{a_n}{b_n}$  converges moderately faster to  $h_q(1)$ .

1.7. For a given prime  $p$ , let  $\text{ord}_p k$  denote the exponent of  $p$  in the prime expansion of  $k$ . Then we observe that

$$(1.7.1) \quad \text{ord}_p \binom{n}{m}_q \leq \text{ord}_p(q)_n - \text{ord}_p(q)_m.$$

**Note:**

$$(1.7.2) \quad \binom{n+k}{k}_q \binom{k}{m}_q = \binom{n+m}{m}_q \binom{n+k}{k-m}_q.$$

**Claim:** The sequences

$$u_n = a_n(q)_{n+1} \prod_{s=[n/2]}^n (1 - q^s) \quad \text{and} \quad z_n = b_n(q)_{n+1} \prod_{s=[n/2]}^n (1 - q^s)$$

are polynomials in  $q$  with integer coefficients, and moreover

$$(1.7.3) \quad z_n = O\left(q^{19n^2/8}\right).$$

**Proof:** Applying (1.7.1) and (1.7.2), we estimate the denominator of  $u_n$ :

$$\begin{aligned} \text{ord}_p \left( \frac{(q^m - 1)(q)_n \binom{n+m}{m}_q}{\binom{n+k}{k}_q} \right) &\leq \text{ord}_p \left( \frac{(q^m - 1)(q)_n \binom{k}{m}_q}{\binom{n+k}{k-m}_q} \right) \\ &\leq \text{ord}_p(q)_n + \text{ord}_p(q^m - 1) + \text{ord}_p(q)_k - \text{ord}_p(q)_m \\ &\leq \text{ord}_p(q)_n + \text{ord}_p \prod_{s=[n/2]}^n (1 - q^s) + \text{ord}_p(q)_k - \text{ord}_p(q)_m \\ &\leq \text{ord}_p \left( (q)_n \prod_{s=[n/2]}^n (1 - q^s) \right), \end{aligned}$$

since  $m \leq k \leq n$ . This proves the claim on  $u_n$ . And (1.7.3) follows from (1.6.1). The rest is trivial.

**Claim:**  $h_q(1) - \frac{u_n}{z_n} = O\left(\frac{1}{z_n^{1+\delta}}\right)$ ; where  $\delta = 0.26316 \dots > 0$ .

**proof:** From (1.6.1), (1.6.4) and (1.7.3), we gather that

$$h_q(1) - \frac{u_n}{z_n} = O\left(b_n^{-2}\right) = O\left(q^{-3n^2}\right) = O\left(z_n^{-1-(5/19)}\right).$$

Thus, we have proved:

**Theorem 1:** If  $|q| > 1$  is an integer,  $h_q(1)$  is irrational with irrationality measure 4.80.

**Note:** By invoking Theorem 7 ([Z], p.596) with  $\omega$  as in 1.1, we obtain the series acceleration:

$$h_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)(q)_n} \quad \text{and} \quad h_q(1) = \sum_{n=1}^{\infty} \frac{1 - q^n - q^{2n}}{(q^n - 1) \binom{2n}{n}_q (q)_n}.$$

## 2. A scheme for proving the irrationality of $Ln_q(2)$ :

The claims made in subsections 2.1-2.5 below were found using the Maple Package `qEKHAD` accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

2.1. The qWZ 1-form  $\omega$  is:

$$\omega = \frac{(-1)^k}{(1 - q^{k+1})} \frac{(q)_n}{\binom{n+k+1}{k+1}_q (q^2)_n} \left\{ \delta k + \frac{q^{n+1}}{(1 + q^{n+1})} \delta n \right\}.$$

2.2. The choice of the potential  $c(n, k)$  is:

$$c(n, k) = \sum_{m=1}^n \frac{q^m (q)_m}{(1 - q^m)(q^2)_m} + \sum_{m=1}^k \frac{(-1)^{m-1}}{(1 - q^m)} \frac{(q)_n}{\binom{n+m}{m}_q (q^2)_n}.$$

2.3. The choice of the mollifier  $b(n, k)$  is:

$$b(n, k) = q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q.$$

2.4. We define two sequences:

$$a(n) = \sum_{k=0}^n c(n, k) b(n, k), \quad \text{and} \quad b(n) = \sum_{k=0}^n b(n, k).$$

2.5. Introduce  $L = y_2(n)N^2 + y_1(n)N + y_0(n)$  and  $B(n, k) = P_q^1(n, k)b(n+1, k)$ , where

$$A(n, k) = c(n, k)B(n, k) + \frac{(-1)^k q^{2n+3}}{1 - q^{n+1}} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}} (q)_{n+1}}{(q^2)_{n+1}} P_q^2(n, k) \quad \text{and}$$

$$P_q^1(n, k) = q\alpha_n^2 [q^3\alpha_n^5 + q^2(1+q)\alpha_n^4 + 2q(1+q^2)\alpha_n^3 - (1-q+q^2)\alpha_n - 3(1+q)] \\ + q\alpha_n^2 [q\beta_k^{-1}(q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2-q)\alpha_n - 2) + (q\alpha_n^3 + (q-1)\alpha_n^2 + (2q-1)\alpha_n)\alpha_k - 2]$$

$$P_q^2(n, k) = q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2+q)\alpha_n + 2 - \alpha_n\alpha_k^2 [\alpha_n^3 + (1-q^{-1})\alpha_n^2 + (2-q)\alpha_n - 2q^{-1}] \\ - \alpha_k [q^2\alpha_n^6 + q(1+q)\alpha_n^5 + (2+q+2q^2)\alpha_n^4 + (1+q)\alpha_n^3 + 2\alpha_n^2 - (2+q+q^{-1})\alpha_n + (q^{-1} - 1)],$$

$$y_0(n) = -q(\alpha_n - 1)(\alpha_n + 1)(q^2\alpha_n^2 + q\alpha_n + 2), \quad y_2(n) = -(q\alpha_n - 1)(q\alpha_n + 1)(\alpha_n^2 + \alpha_n + 2),$$

$$y_1(n) = q^4\alpha_n^7 + q^2(1+q)(q\alpha_n^6 + \alpha_n^4) + q(1+q+q^2)(2q\alpha_n^5 + \alpha_n^3) - (1+3q+3q^2+q^3)\alpha_n^2 - (1+q^2)(2+\alpha_n),$$

$$\text{and } \alpha_n = q^{n+1}, \quad \beta_k = q^{k+1}.$$

Then

$$(**) \quad L(b(n, k)) = B(n, k) - B(n, k-1) \quad \text{and} \quad L(b(n, k)c(n, k)) = A(n, k) - A(n, k-1).$$

Now, summing over  $k$  in  $(**)$  shows that both sequences  $a(n)$  and  $b(n)$  are solutions of  $Lu(n) = 0$ .

2.6. Similar arguments and estimates as in (1.6) above lead to

$$(2.6.1) \quad Ln_q(2) - \frac{a_n}{b_n} = O(b_n^{-2}).$$

In particular, the sequence of rational numbers  $\frac{a_n}{b_n}$  converges moderately faster to  $Ln_q(2)$ .

**2.7. Claim:** The sequences

$$u_n = a_n \prod_{t=1}^n (1 + q^t) \prod_{s=[n/2]}^n (1 - q^s) \quad \text{and} \quad z_n = b_n \prod_{t=1}^n (1 + q^t) \prod_{s=[n/2]}^n (1 - q^s)$$

are polynomials in  $q$  with integer coefficients, and moreover

$$(2.7.1) \quad z_n = O(q^{19n^2/8}).$$

**Proof:** Applying (1.7.2) and (2.7.1), we estimate the denominator of  $u_n$ :

$$\begin{aligned} ord_p \left( \frac{(1 - q^m)(q^2)_n \binom{n+m}{m}_q}{\binom{n+k}{k}_q (q)_n} \right) &\leq ord_p \left( \frac{(q^m - 1)(q^2)_n \binom{k}{m}_q}{\binom{n+k}{k-m}_q (q)_n} \right) \\ &\leq ord_p \left( \frac{(q^2)_n}{(q)_n} \right) + ord_p(q^m - 1) + ord_p(q)_k - ord_p(q)_m \\ &\leq ord_p \left( \frac{(q^2)_n}{(q)_n} \right) + ord_p \prod_{s=[n/2]}^n (1 - q^s) + ord_p(q)_k - ord_p(q)_m \\ &\leq ord_p \left( \prod_{t=1}^n (1 + q^t) \prod_{s=[n/2]}^n (1 - q^s) \right), \end{aligned}$$

since  $m \leq k \leq n$ . This proves the claim on  $u_n$ . And (2.7.1) follows from (1.6.1). The rest is trivial.

**Claim:**  $Ln_q(2) - \frac{u_n}{z_n} = O\left(\frac{1}{z_n^{1+\delta}}\right)$ ; where  $\delta = 0.26316 \dots > 0$ .

**proof:** From (1.6.1), (2.6.1) and (2.7.1), we gather that

$$Ln_q(2) - \frac{u_n}{z_n} = O(b_n^{-2}) = O(q^{-3n^2}) = O(z_n^{-1-(5/19)}).$$

Thus, we have proved:

**Theorem 2:** If  $|q| \neq 0, 1$  is an integer,  $Ln_q(2)$  is irrational with irrationality measure 4.80.

**Note:** By invoking Theorem 7 ([Z], p. 596) with  $\omega$  as in 2.1, we obtain the series acceleration:

$$Ln_q(2) = \sum_{n=1}^{\infty} \frac{q^n (q)_n}{(1 - q^n)(q^2)_n} \quad \text{and} \quad Ln_q(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (q)_n (1 - q^{3n})}{(1 - q^n)^2 \binom{2n}{n}_q (q^2)_n}.$$

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The package qEKHAD is available by the www at <http://www.math.temple.edu/~zeilberg/programs.html>
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