

A PARKING FUNCTION ANALOG OF THE SCHRÖDER NUMBERS

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ABSTRACT. Pattern avoidance has played a central role in the enumeration of permutations, with many classical sequences arising from restrictions on the patterns that may occur. Parking functions provide a natural setting in which to ask analogous questions. In this paper, we study parking functions whose parking permutations avoid the patterns 1234 and 2134, obtaining an analog of the Schröder numbers for parking functions. To enumerate these objects, we develop an enumeration scheme that runs in polynomial time and adapts the prefix-based framework introduced by Zeilberger in 1998 by instead using “suffixes,” following an approach developed by Vatter.

1. INTRODUCTION

Enumerating pattern avoiding permutation classes has a long history. In 1985, Simion and Schmidt enumerated permutations avoiding a pair of patterns of length 3. Many further enumerative results have followed for patterns in permutations and in words, see Bóna [6], Kitaev [9] and Vatter [17] for surveys. Numerous systematic techniques have been developed for enumerating permutation classes. These include generating trees [7, 16, 18, 19], insertion encoding [4], substitution decompositions [5], and enumeration schemes [21].

In 1998, Doron Zeilberger introduced prefix enumeration schemes to count pattern avoiding permutations by providing an automated method for counting Wilf equivalence classes [21]. However, the success rate for finding these schemes was limited. In 2005, Vatter extended Zeilberger’s results by introducing “gap vectors”, which completely automated the enumeration of many more Wilf classes making the enumeration scheme much more powerful [15]. Later, Zeilberger reformulated Vatter’s schemes back to his original notation, allowing for even quicker enumeration of many pattern avoiding permutation classes [22]. Other work on enumeration schemes includes that of Pudwell, who studied pattern avoidance in words and barred permutations by automating new enumeration results [12, 13]. More recently, a new (systematic) algorithmic framework was introduced called combinatorial exploration, which automatically and rigorously analyzes the structure of combinatorial objects and derives their counting sequences and generating functions [3].

A superset of permutations are called parking functions. Parking functions can be described by means of a deterministic parking process. Consider a parking lot with n parking spots on a one-way street labeled 1 to n . A sequence of n cars enters the street one by one, from left to right, with car i having a preferred spot a_i , which we call its parking preference. For each $i \in [n]$, car i drives to its preferred spot a_i and attempts to park. If the spot is not available, the car continues to drive down the street and parks in the next available spot, if one exists. If there is no available spot, the car exits the lot and is unable to park. We call this the parking rule, and the sequence α is called a *parking function* if all cars can park under the parking rule. Konheim and Weiss established that there are $(n + 1)^{n-1}$ parking functions of length n [10]. For a parking function α , the order in which the cars park on the street is called the *parking permutation* and is denoted by $\mathcal{O}(\alpha)$. If $\mathfrak{S}_n$ denotes the set of permutations of $[n]$ written in one-line notation, then the parking permutation of α is

$$\mathcal{O}(\alpha) = \pi_1 \pi_2 \cdots \pi_n,$$

where π_i denotes that car π_i parked in spot i on the street.

Since there is a many-to-one correspondence between parking functions and permutations, it is natural to study pattern avoidance in parking functions. Qiu and Rempel introduced a notion of pattern avoidance for parking functions by viewing them as labeled Dyck paths [14]. Adeniran and Pudwell [2] further studied this notion and posed several open problems, which were subsequently resolved by Adenbaum [1]. Yan introduced a second notion of pattern avoidance in parking functions by studying the parking permutations [20]. These parking permutations record the order in which the cars park on the street. Yan enumerated all parking functions whose parking permutations avoid any collection of patterns of length 3.

In this paper, we use Yan's notion of pattern avoidance. We use enumeration schemes to systematically enumerate parking functions whose parking permutations avoid the patterns 1234 and 2134, and obtain a recurrence that can be computed in polynomial time. In the Wilf sense, the corresponding permutation class are enumerated by the Schröder numbers [11]. This paper is organized as follows. In Section 2, we introduce the necessary background on parking functions and pattern avoidance. In Section 3, we reformulate the prefix enumeration scheme in terms of suffixes and extend it to parking enumeration schemes. We illustrate this approach by recovering a few of Yan's results [20] for patterns of length 3. In Section 4, we apply the resulting framework to the patterns 1234 and 2134, and we obtain our main enumeration of parking functions whose parking permutations avoid these patterns.

2. BACKGROUND

2.1. Parking Functions. Colmenarejo et al. [8] provide a technique for enumerating parking functions by counting through permutations. We restate their result below and use Lemma 2.1 to construct sequences (Definition 2.1), which we refer to as parking tables, which play a role in the main result.

Lemma 2.1. Given a permutation $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, define

$$\ell(\pi_i) = \max\{k : \pi_j \leq \pi_i \text{ for all } i - k + 1 \leq j \leq i\}.$$

The number of parking functions with parking permutation π is

$$\prod_{i=1}^n \ell(\pi_i).$$

Example 1. Let $n = 5$ and $\pi = 23154$ be the permutation that parks the cars on the street. Then, $\ell(\pi_1) = 1, \ell(\pi_2) = 2, \ell(\pi_3) = 1, \ell(\pi_4) = 4$ and $\ell(\pi_5) = 1$. Moreover, the product of $1 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 8$ is the total number of parking functions with parking permutation $\pi = 23154$.

Definition 2.1. Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$. We construct the sequence $p = (p_1, p_2, \dots, p_n) \in [n]^n$, which we call the *parking table*, so that for all i ,

$$p_i = \ell(\pi_i),$$

as defined in Lemma 2.1. We will often write p in one-line notation.

2.2. Pattern Avoidance. For $m \leq n$ and $\sigma \in \mathfrak{S}_m, \pi \in \mathfrak{S}_n$, we say that π *contains* σ as a pattern if there exists $1 \leq i_1 < \cdots < i_m \leq n$, such that $\pi(i_a) < \pi(i_b)$ if and only if $\sigma(a) < \sigma(b)$ for all $a, b \in [m]$, and we say π *avoids* σ otherwise. For any collection $\sigma_1, \dots, \sigma_k$ of permutations, $\text{Av}_n(\sigma_1, \dots, \sigma_k)$ denotes the set of all permutations in $\mathfrak{S}_n$ containing none of the permutations $\sigma_1, \dots, \sigma_k$ as a pattern. For two permutations $\sigma, \tau \in \mathfrak{S}_k$, we say that σ and τ are *Wilf equivalent* if $|\text{Av}_n(\sigma)| = |\text{Av}_n(\tau)|$ for all $n \geq k$.

Definition 2.2. For a collection of permutations $\sigma_1, \dots, \sigma_k$, let $\text{Pk}_n(\sigma_1, \dots, \sigma_k)$ be the set of parking functions α such that its associated parking permutation π contains none of $\sigma_1, \dots, \sigma_k$ as a pattern and let $\text{pk}_n(\sigma_1, \dots, \sigma_k) = |\text{Pk}_n(\sigma_1, \dots, \sigma_k)|$.

An equivalent characterization of pattern containment is given in terms of “reduction” of words. This perspective is particularly useful in our later enumeration arguments for the parking analogs of the Schröder numbers.

Definition 2.3. Let $[k]^n$ denote the set of words of length n in the alphabet $\{1, 2, \dots, k\}$. Let $w \in [k]^n$ and $w = w_1 w_2 \cdots w_n$. The reduction of w , denoted $\text{red}(w)$, is the unique word of length n obtained by replacing the i^{th} smallest distinct letter of w with i , for each i . We refer to $\text{red}(w)$ as the *reduced permutation*.

Example 2. If $\pi = 23154$, deleting the entry $\pi_3 = 1$ yields the word 2354, whose reduced permutation is $\text{red}(2354) = 1243$.

The following equivalent definition of pattern containment will be convenient.

Definition 2.4. Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$ and let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in \mathfrak{S}_k$. We say that π contains σ if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that $\text{red}(\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}) = \sigma$. Otherwise π avoids σ .

Claim 1. Let $k \leq n \in \mathbb{N}$ and Π be a collection of patterns. If $\pi \in \text{Av}_n(\Pi)$ then $\text{red}(w) \in \text{Av}_k(\Pi)$ for any subsequence w of π of length k .

Notice that the parking table sequences can be defined for words with distinct values. If w is a word with distinct values and parking table p , then its reduced permutation also yields the same parking table p .

Claim 2. Let $w \in [k]^n$ with distinct elements and $\sigma \in \mathfrak{S}_n$. If $\sigma = \text{red}(w)$ for a word w with distinct elements then the parking tables of w and σ coincide.

Example 3 (Continuing Example 2). The parking table of the word 2354 is $p = (1, 2, 3, 1)$ which is the same parking table of its reduced permutation 1243.

Definition 2.5. Let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ be a parking permutation and $p = p_1 p_2 \cdots p_n \in [n]^n$ be its parking table. For $1 \leq k \leq j \leq n$, we call a pair (τ, p') a *bi-suffix* of π if

$$\tau = \pi_k \pi_{k+1} \cdots \pi_j \in [n]^{j-k+1}$$

and

$$p' = p_k p_{k+1} \cdots p_j \in [n]^{j-k+1}.$$

Example 4. Let $\pi = 2471635 \in \mathfrak{S}_7$ and $k = 4, j = 7$. The parking table of π is given by $p = 1231212$. Then the bi-suffix of π is the tuple (τ, p') where $\tau = 1635$ and $p' = 1212$.

Lemma 2.2. Given $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ with bi-suffix (i, r) , where $\pi_n = i$ and $\ell(\pi_n = i) = r$. Let $w \in [n]^{n-1}$ be the word obtained by deleting $\pi_n = i$ and $\text{red}(w) = \tau = \tau_1 \cdots \tau_{n-1} \in \mathfrak{S}_{n-1}$. Then

$$\prod_{i=1}^n \ell(\pi_i) = r \prod_{i=1}^{n-1} \ell(\tau_i).$$

In other words, the number of parking functions with parking permutation π is the same as the product of r and the number of parking functions with parking permutation τ .

Proof. This follows by applying Lemma 2 to $w = \pi_1 \pi_2 \cdots \pi_{n-1}$ and $\text{red}(w) = \tau$. Since the parking tables of both w and τ are the same, then

$$\prod_{i=1}^{n-1} \ell(\tau_i) = \prod_{i=1}^{n-1} w_i = \prod_{i=1}^{n-1} \ell(\pi_i).$$

Thus,

$$\prod_{i=1}^n \ell(\pi_i) = \ell(\pi_n) \prod_{i=1}^{n-1} \ell(\tau_i) = r \prod_{i=1}^{n-1} \ell(\tau_i).$$

□

3. FROM PERMUTATION ENUMERATION SCHEMES TO PARKING ENUMERATION SCHEMES

We review the scheme first described by Zeilberger [21] and later improved by Vatter [15]. We do not go into the details of the enumeration scheme, but we follow the notation used in Zeilberger's survey [22]. The goal of an enumeration scheme is to describe a class of avoiding permutations in terms of smaller permutations in the class via recursion. The original scheme method used prefixes. We implement Vatter's Maple package so that we can use suffixes, which is available in [?]. Using suffixes allows us to analyze the last elements of the permutation along with the last elements of the corresponding parking table.

For this section only, we need the following definition.

Definition 3.1. Let $\Pi = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ be collection of patterns. For $m \geq 1$, let $B_n^{(m)}(\Pi)$ denote the set of all bi-suffixes (τ, p') , with $\tau, p' \in [n]^m$, that arise from permutations $\pi \in \text{Av}_n(\Pi)$ and its corresponding parking table $p \in [n]^n$. That is,

$$B_n^{(m)}(\Pi) = \{(\tau, p') : (\tau, p') \text{ is a bi-suffix of } \pi \in \text{Av}_n(\Pi) \text{ with parking table } p\}.$$

We refer to τ and p' as the permutation component and parking table component, respectively.

For each $(\tau, p') \in B_n^{(m)}(\Pi)$, we define

$$B_n^{(m+1)}(\Pi; (\tau, p')) = \left\{ (\mu, q) : \begin{array}{l} (\mu, q) \text{ is a bi-suffix of } \pi \in \text{Av}_n(\Pi) \text{ with parking table } p, \\ \mu_2 \cdots \mu_{m+1} = \tau \text{ and } q_2 \cdots q_{m+1} = p' \end{array} \right\}.$$

Warning: We abuse notation whenever $\tau = i$ and $p' = r$ are length 1 and simply write (i, r) instead of $((i), (r))$.

Example 5. Let $n = 4$ and $\pi = 2413 \in \text{Av}_4(123)$ with parking table $p = 1212$. Then, $(3, 2)$ is a member in the set $B_4^{(1)}(123)$ while $((1, 3), (1, 2))$ is a member in the set $B_4^{(2)}(123; (3, 2))$.

We begin with an example of the enumeration scheme for the 123-avoiding class and illustrate how to adapt the enumeration scheme for parking functions by recovering Yan's results [20].

3.1. The 123-avoiding permutation scheme. Let $A(n)$ be the set of 123-avoiding permutations, i.e. $A(n) := \text{Av}_n(123)$. We can describe the set $A(n)$ by taking collections of the permutations fixed by their last element. Precisely, define

$$A_1(n, i) := \{\pi \in A(n) : \pi_n = i\}.$$

Hence,

$$A(n) = \bigcup_{i=1}^n A_1(n, i),$$

which is the union of all permutations that end with the element i for all $i \in [n]$.

Moreover, the last two elements of any 123-avoiding permutation form a word in the alphabet $[n]$ of length 2. If $n \geq 2$, then we can define the following two sets according to the reduction of the last two elements,

$$A_{12}(n, j, i) := \{\pi \in A(n) : \pi_{n-1} = j, \pi_n = i, j < i\},$$

and

$$A_{21}(n, h, i) := \{\pi \in A(n) : \pi_{n-1} = h, \pi_n = i, h > i\}.$$

Here, $A_{12}(n, j, i)$ is the set of all permutations whose last two elements end with ji and $j < i$. Similarly, $A_{21}(n, h, i)$ is the set of all permutations whose last two elements end with hi and $h > i$.

In the enumeration scheme, the goal is to keep describing sets in terms of collections of smaller sets until the enumeration is equivalent for some collections. If $i = 1$ then $A_1(n, i)$ is equinumerous to the set $A(n-1)$ since if $\pi \in A_1(n, i)$ then removing i from π yields a word w of length $n-1$ whose reduction $\text{red}(w)$ is a permutation of length $n-1$. The same argument follows if $i = 2$. In both cases, both reductions avoid the pattern 123 (Claim 1).

If $i \geq 3$, then we have

$$A_1(n, i) = \left(\bigcup_{j=1}^{i-1} A_{12}(n, j, i) \right) \cup \left(\bigcup_{h=i+1}^n A_{21}(n, h, i) \right).$$

We show that $|A_{21}(n, h, i)| = |A_1(n-1, h-1)|$. Let $\pi \in A_{21}(n, h, i)$ then removing the element i from π results in the word $w = \pi_1 \pi_2 \cdots \pi_{n-2} h$ of length $n-1$, and $\text{red}(w)$ replaces every element k in w by $k-1$ whenever $k > i$. Thus, the reduced permutation ends with $h-1$ because $k = h$ and $h > i$. In addition, the reduced permutation avoids the pattern 123 (Claim 1). Thus, the set $A_{21}(n, h, i)$ is equinumerous with the set $A_1(n-1, h-1)$ for $i+1 \leq h \leq n$, or $|A_{21}(n, h, i)| = |A_1(n-1, h)|$ with $i \leq h \leq n-1$.

For the set $A_{12}(n, j, i)$, the suffix scheme relies on *gap vectors*. Gap vectors give a condition for which choices of elements in the suffix yield empty permutations. Precisely, if $\pi = \pi_1 \pi_2 \cdots \pi_j i_1 i_2 i_3 \cdots i_{n-j}$ then a gap vector provides information on the possible values of the elements $i_1, i_2, \dots, i_{n-j}$. In our example, we obtain the information that if $A_{12}(n, j, i) \neq \emptyset$ if and only if $j = 1$. Hence, if $j = 1$ and $\pi \in A_{12}(n, j, i)$, then removing the element $j = 1$ from π results in the word $w = \pi_1 \pi_2 \cdots i$ of length $n-1$, and $\text{red}(w)$ replaces every element k in w by $k-1$ whenever $k > j$. Thus, the reduced permutation ends with $i-1$ because $k = i$ and $i > j = 1$. In addition, the reduced permutation avoids the pattern 123 (with a similar argument as before). Thus, the set $A_{12}(n, j, i)$ is equinumerous to $A_1(n-1, i-1)$.

Therefore, if $i \geq 3$, then

$$|A_1(n, i)| = \sum_{h=1}^{n-1} |A_1(n-1, h)| + |A_1(n-1, i-1)|,$$

otherwise $i = 1$ or $i = 2$ then $|A_1(n, i)| = |A(n-1)|$. Since the enumeration scheme for the 123-avoiding permutations only checked up to the last two elements in the permutation, we say that the enumeration scheme has depth 2.

Proposition 3.1. Let $a(n)$ be the number of 123-avoiding permutations and $a(n, i)$ be those permutations that end with the element i . Then

$$a(n, i) = \begin{cases} a(n-1) & \text{if } i = 1 \text{ or } i = 2 \\ a(n-1, i-1) + \sum_{j=i}^{n-1} a(n-1, j) & \text{otherwise} \end{cases}$$

with initial conditions $a(1, 1) = 1$ and $a(1, i) = 0$ if $i \geq 2$.

3.2. The 123-avoiding parking scheme. The goal of the parking enumeration scheme is to determine how to recurrence to the enumeration scheme for the avoiding permutation families changes. A natural question is whether there exists constants coefficients, depending on the values n and i , for the recurrence in Proposition 3.1.

We begin with a characterization of the suffixes for the 123-avoiding permutations.

Lemma 3.2. The set of bi-suffixes in $B_n^{(1)}(123)$ is given by

$$B_n^{(1)}(123) = \{(i, r) : 1 \leq r \leq i \leq n-1\} \cup \{(n, n)\}.$$

In other words, if a 123-avoiding parking permutation ends in $i < n$, then the corresponding parking table can end in any $r \in [i]$, while if it ends in n , the parking table necessarily ends in n .

Proof. For any permutation $\pi \in \text{Av}_n(123)$, the last element $\pi_n = i$ can range from any value from 1 to n .

If $\pi_n = n$ then all elements to the left of π_n are less than n so $\ell(\pi_n) = p_n = n$ in the parking table. Hence, $(n, n) \in B_n^{(1)}(123)$. If $\pi_n = i$ and $i < n$, by definition, if $\ell(\pi_n) = r$ then there are $r-1$ elements to the left of π_n that are all at most i . Since there only i elements of $[n]$ that are at most i , we have $r \leq i$. Thus, $(i, r) \in B_n^{(1)}(123)$ for $1 \leq r \leq i \leq n-1$.

We now assume $1 \leq r \leq i \leq n-1$ and construct a 123-avoiding permutation whose last element $\pi_n = i$ and whose last entry in the parking table $p_n = r$. Consider

$$\pi = (i-1)(i-2) \cdots rn(n-1) \cdots (i+1)(r-1)(r-2) \cdots 1i.$$

Notice that π can be decomposed into four blocks. The first block B_1 contains the elements $r, r+1, \dots, i-2, i-1$. If $i = r$, then B_1 is an empty block, otherwise, if $r < i$ then there are $i-r$ elements in block B_1 in decreasing order. The second block B_2 contains the elements $i+1, i+2, \dots, n-1, n$ in decreasing order, and there are $n-i$ of them. The third block B_3 contains the elements $1, 2, \dots, r-2, r-1$ in decreasing order and there are $r-1$ of them. The last block B_4 is the last element i . Altogether, there are n elements in the permutation π .

Furthermore, $\ell(\pi_n) = r$ because the first element j to the left of i such that $j > i$ is at position $n-r$. Mainly, $\pi_{n-r} = i+1$ and $\pi_{n-r} > i$. For any $n-r+1 \leq k \leq n-1$, $\pi_k \leq r-1 < i$ and so $p_n = \ell(\pi_n) = r$. Moreover, π avoids the pattern 123. Indeed, elements in blocks B_1 , B_2 and B_3 are decreasing. Also, for any $b_1 \in B_1, b_2 \in B_2$ and $b_3 \in B_3$, $b_3 < b_1 < b_2$ and $b_1 < i < b_2$. Thus, there are no increasing subsequences as $b_3 < b_1 < i < b_2$. $\square$

Example 6. For $n = 3$, there are five 123-avoiding permutations, and for each, we list the bi-suffixes below.

Permutation π	Parking table p	(i, r) where $i = \pi_3$ and $r = p_3$
132	121	(2, 1)
213	113	(3, 3)
231	121	(1, 1)
312	112	(2, 2)
321	111	(1, 1)

Hence, $B_3^{(1)}(123) = \{(1, 1), (2, 1), (2, 2), (3, 3)\}$.

It turns out that we can track down the last two elements of the parking permutation and parking table provided that the last elements of both are known. By tracking down the last two elements of the parking permutation and parking table, we can obtain a recurrence that counts the number of parking functions whose parking permutations avoid the pattern 123.

Lemma 3.3. Given $(i, r) \in B_n^{(1)}(123)$, the set of pairs (j, s) for which

$$((j, i), (s, r)) \in B_n^{(2)}(123; (i, r))$$

is characterized as follows:

$$\{(j, s) : ((j, i), (s, r)) \in B_n^{(2)}(123; (i, r))\} = \begin{cases} \{(j, s) : 1 \leq s < j, i < j < n\} \cup \{(n, n-1)\} & \text{if } r = 1 \\ \{(1, 1)\} & \text{otherwise.} \end{cases}$$

Proof. We first prove that if $(i, r) \in B_n^{(1)}(123)$ then $(\tau, p') \in B_n^{(2)}(123; (i, r))$ is of the desired form.

Let $\pi = \pi_1 \pi_2 \cdots \pi_{n-2} j i \in \text{Av}_n(123)$ and $p = (p_1, p_2, \dots, p_{n-2}, s, r)$ be its parking table. By the suffix scheme from Section 3.1, if π ends with ji then either $j > i$ or $j < i$.

If $j > i$ then $\ell(\pi_n = i) = 1$ and so $r = 1$ in the parking table p . We have the following cases:

- (1) Suppose that $j = n$. Then, $\pi = \pi_1 \pi_2 \cdots \pi_{n-2} n i$ and $s = \ell(\pi_{n-1} = n) = n - 1$ since for all $1 \leq k \leq n - 2$, $\pi_k < n$. Thus, $\{((n, i), (n - 1, 1))\} \in B_n^{(2)}(123; (i, r))$ whenever $r = 1$ and $\pi_{n-1} = n, \pi_n = i$.
- (2) If $j < n$ then $i + 1 \leq j \leq n - 1$ as $j > i$. We show that $1 \leq s = \ell(\pi_{n-1} = j) \leq j - 1$. If $s = \ell(\pi_{n-1} = j)$ then the only possible elements S contiguously to the left of $\pi_{n-1} = j$ such that $1 \leq s \leq j - 1$ are if $S = \{1, 2, \dots, i - 1\} \cup \{i + 1, i + 2, \dots, j - 1\}$, where $|S| = j - 2$. Hence, $1 \leq s \leq j - 2 + 1 = j - 1$. It follows that $\{(\tau, p') : 1 \leq s \leq j - 1, i + 1 \leq j \leq n - 1\} \in B_n^{(2)}(123; (i, r))$ whenever $r = 1$ and $\pi_{n-1} < n, \pi_n = i$.

If $j < i$ then $r = \ell(\pi_n = i) > 1$. But $j = 1$ by the suffix scheme from Section 3.1. Therefore, $\pi_{n-2} > \pi_{n-1} = 1$. It follows that $s = \ell(\pi_{n-1} = 1) = 1$. Thus, $\{((1, i), (1, r))\} \in B_n^{(2)}(123; (i, r))$ whenever $\pi_{n-1} = j, \pi_n = i$ with $j < i$ and $p_n = r > 1$.

Now, we construct a $\pi \in B_n^{(2)}(123; (i, r))$ such that $\pi \in \text{Av}_n(123)$. If $r = 1$ then construct π such that

$$\pi = \underbrace{(n-1)(n-2) \cdots (i+1)}_{n-i-1} \underbrace{(i-1)(i-2) \cdots 321}_{i-1} n i$$

or

$$\pi = \underbrace{n(n-1) \cdots (j+2)(j+1)}_{n-j} \underbrace{(j-1)(j-2) \cdots (i+1)}_{j-i-1} \underbrace{(i-1)(i-2) \cdots 321}_{i-1} j i$$

with $j > i$. Both permutations are 123-avoiding and both satisfy the condition that if $\pi_{n-1} = j$ then either $j = n$ or $i < j < n$ with $s = \ell(\pi_{n-1}) = n - 1$ or $1 \leq s = \ell(\pi_{n-1} = j) \leq j - 2 + 1 = j - 1$, respectively. $\square$

Example 7. For $n = 4$, there are fourteen 123-avoiding permutations, and there are seven bi-suffixes where the permutation and parking table components are of length 1. For each of those bi-suffixes, we list the bi-suffixes where the permutation and parking table components are of length 2.

Bi-suffix (i, r)	Bi-suffix $((j, i), (s, r))$	Permutation π	Parking table p
(1, 1)	$((2, 1), (1, 1))$	3421	1211
		4321	1111
	$((3, 1), (1, 1))$	2431	1211
	$((3, 1), (2, 1))$	4231	1121
(2, 1)	$((4, 1), (3, 1))$	3241	1131
	$((3, 2), (1, 1))$	1432	1211
	$((3, 2), (2, 1))$	4132	1121
	$((4, 2), (3, 1))$	3142	1131
(2, 2)	$((1, 2), (1, 2))$	3412	1212
		4312	1112
(3, 1)	$((4, 3), (3, 1))$	2143	1131
(3, 2)	$((1, 3), (1, 2))$	2413	1212
(3, 3)	$((1, 3), ((1, 3)))$	4213	1113
(4, 4)	$((1, 4), (1, 4))$	3214	1114

Proposition 3.4. Let $a(n)$ be the number of parking functions whose parking permutations avoid 123. For each bi-suffix $S = (i, r) \in B_n^{(1)}(123)$, let $a(n, S)$ denote the number of parking functions whose parking permutations π have bi-suffix S . Then

$$a(n) = \sum_{S \in B_n^{(1)}(123)} a(n, S),$$

with initial conditions $a(1, (1, 1)) = 1$ and $a(1, S) = 0$ otherwise.

Fix $S = (i, r) \in B_n^{(1)}(123)$. For each pair (j, s) such that

$$((j, i), (s, r)) \in B_n^{(2)}(123; (i, r)),$$

let $S' = ((j, i), (s, r))$ and $a(n, S, S')$ denote the number of parking functions whose parking permutations π have bi-suffix S' . Then

$$a(n, S) = \sum_{\substack{(j, s) \text{ such that} \\ ((j, i), (s, r)) \in B_n^{(2)}(123; (i, r))}} a(n, S, S').$$

Furthermore,

$$a(n, S, S') = a(n, (i, r), (j, s)) = \begin{cases} a(n-1, (j-1, s)), & \text{if } r = 1 \text{ and } j > i, \\ \frac{r}{r-1} a(n-1, (i-1, r-1)), & \text{if } r > 1 \text{ and } j = s = 1. \end{cases}$$

Proof. The proof involves deleting exactly the elements given by the suffix permutation scheme and reducing the permutation to a smaller permutation. For each permutation, the recurrence admits the suffixes are characterized in Lemma 3.3. We omit the proof as it follows from the previous section. $\square$

4. MAIN RESULT

In the previous section, we described the suffix enumeration scheme that is extendable to pattern avoidance in parking functions. We now present the recurrence that provides the number of parking functions whose parking permutations avoid the patterns 1234 and 2134. We begin by describing the enumeration scheme for the $\{1234, 2134\}$ -avoiding permutations and then use it for the parking functions setting.

4.1. The permutation scheme. The enumeration scheme for the $\{1234, 2134\}$ -avoiding permutations is similar to the scheme for the 123-avoiding permutations, described in Section 3.1. We describe it here but leave out some of the details.

Let $A(n) = \text{Av}_n(1234, 2134)$ and define

$$A_1(n, i) := \{\pi \in A(n) : \pi_n = i\}.$$

Hence,

$$A(n) = \bigcup_{i=1}^n A_1(n, i).$$

The enumeration scheme for the permutations in the set $A(n)$ is depth 2. For $n \geq 2$, define

$$A_{12}(n, j, i) := \{\pi \in A(n) : \pi_{n-1} = j, \pi_n = i, j < i\},$$

and

$$A_{21}(n, h, i) := \{\pi \in A(n) : \pi_{n-1} = h, \pi_n = i, h > i\}.$$

As in the 123-avoiding permutations case, if $i = 1$ or $i = 2$ then $A_1(n, i)$ is equinumerous to the set $A(n-1)$. If $i \geq 3$, then we have

$$A_1(n, i) = \left(\bigcup_{j=1}^{i-1} A_{12}(n, j, i) \right) \cup \left(\bigcup_{h=i+1}^n A_{21}(n, h, i) \right).$$

Following similar arguments discussed in Section 3.1, we obtain $|A_{21}(n, h, i)| = |A_1(n-1, h-1)|$. For the set $A_{12}(n, j, i)$, the scheme relies on gap vectors. It turns out that $A_{12}(n, j, i) \neq \emptyset$ if and only if $j = 1$ or $j = 2$. For any $j > 2$, $|A_{12}(n, j, i)| = 0$. Hence, if $j = 1$ then $|A_{12}(n, j, i)| = |A_1(n-1, i-1)|$. Similarly, if $j = 2$ then $|A_{12}(n, j, i)| = |A_1(n-1, i-1)|$.

Therefore, if $i \geq 3$, then

$$|A_1(n, i)| = \sum_{h=1}^{n-1} |A_1(n-1, h)| + 2|A_1(n-1, i-1)|,$$

otherwise if $i = 1$ or $i = 2$ then $|A_1(n, i)| = |A(n-1)|$.

Proposition 4.1. Let $a(n)$ be the number of 1234, 2134-avoiding permutations and $a(n, i)$ be those permutations that end with the element i . Then

$$a(n, i) = \begin{cases} a(n-1) & \text{if } i = 1 \text{ or } i = 2 \\ 2a(n-1, i-1) + \sum_{j=i}^{n-1} a(n-1, j) & \text{otherwise} \end{cases}$$

with initial conditions $a(1, 1) = 1$ and $a(1, i) = 0$ if $i \geq 2$.

4.2. The parking enumeration scheme. We begin with a claim that is used in proof of the main result. Recall that if $\pi \in \text{Av}_n(1234, 2134)$ and $\pi_{n-1} < \pi_n$ then $\pi_{n-1} = 1$ or $\pi_{n-1} = 2$.

Lemma 4.2. Let $\pi \in \text{Av}_n(1234, 2134)$ and $\pi_{n-1} < \pi_n$. Let $k \in [n-2]$ and $\pi_k = 1$. If $\pi_{n-1} = 2$ and $\pi_n = n$ then

$$(1, \pi_{k+1}, \pi_{k+2}, \dots, \pi_{n-2}, 2, n) = (1, \underbrace{r_1, r_1 - 1, \dots, 3}_{r_1-2}, 2, n),$$

whenever $(1, \pi_{k+1}, \pi_{k+2}, \dots, \pi_{n-2}, 2)$ is of length r_1 , $2 \leq r_1 \leq n-1$. As a consequence, the parking table of $(1, \underbrace{r_1, r_1 - 1, \dots, 3}_{r_1-2}, 2)$ is $(1, 2, \underbrace{1, \dots, 1}_{r_1-2})$.

Proof. Let $(1, \pi_{k+1}, \pi_{k+2}, \dots, \pi_{n-2}, 2)$ be of length r_1 , $0 \leq r_1 \leq n-1$. Assume that $\pi_a > r_1$ for some $a, k < a \leq n-2$. Then there exists $b \leq k$ such that $\pi_b < r_1$ such that π_b is to the left of $\pi_k = 1$. It follows that $\pi_b, \pi_k = 1, \pi_a, n$ is order isomorphic to 2134 since $1 = \pi_k < \pi_b < \pi_a < n$, a contradiction. $\square$

Lemma 4.3. Let $\pi \in \text{Av}_n(1234, 2134)$ and $\pi_{n-1} < \pi_n$. Let $i \in [n]$, $r \in [i-1]$, and $r < i$. Let $\pi_n = i, \pi_{n-1} = 2$ and $\pi_a = 1$ for some $a \leq n-r$ such that $\pi_k < \pi_n = i$ for all $n-r+1 \leq k \leq n-2$ and $\pi_{n-r} > \pi_n = i$. Then

$$(\pi_{n-r+1}, \pi_{n-r+2}, \dots, \pi_{n-2}, 2, i) = (r, r-1, \dots, 3, 2, i).$$

Proof. Let $\pi_a = 1$ for some $a \leq n-r$, and $\pi_n = i, \pi_{n-1} = 2$ such that $\pi_k < \pi_n = i$ for all $n-r+1 \leq k \leq n-2$. By assumption, there are $r-2$ positions between π_{n-r} and $\pi_{n-1} = 2$ and the elements at those positions must be from the set $\{3, 4, \dots, r+1, \dots, i-1\}$. Suppose that there exists at least one $b > n-r$ such that $i > \pi_b > r$. This implies that there is at least one $c \in [n-r]$ such that $\pi_c < r$ as there are $r-2$ elements in the set $\{3, 4, \dots, r\}$ and only $r-3$ possible positions to the right of π_{n-r} . There are two cases,

either π_c is somewhere to the left of $\pi_a = 1$ or π_c is somewhere to the right of $\pi_a = 1$ (and before π_{n-r+1}). If π_c is to the left of $\pi_a = 1$ then the subsequence $\pi_c < r, \pi_a = 1, \pi_b > r, \pi_n = i$ is order isomorphic to 2134 as $\pi_a = 1 < \pi_c < r < \pi_b < \pi_n = i$. Otherwise, π_c is to the right of $\pi_a = 1$ then the subsequence $\pi_a = 1, \pi_c < r, \pi_b > r, \pi_n = i$ is order isomorphic to 1234. In either case, both lead to a contradiction as π is $\{1234, 2134\}$ -avoiding. In addition, the elements $\pi_{n-r+1}, \pi_{n-r+2}, \dots, \pi_{n-2}$ must be in decreasing order because if $\pi_d < \pi_e$ for $n-r+1 \leq d < e \leq n-2$, then the subsequence $\pi_a = 1, \pi_a, \pi_b, \pi_n = i$ forms a 1234 pattern. Thus, the result follows. $\square$

We recall the following definitions and notation.

Definition 4.1. Let $n \in \mathbb{N}$.

- (1) $\text{Av}_n(1234, 2134)$ is the set of permutations of length n that avoid the patterns 1234 and 2134.
- (2) $\text{Pk}_n(1234, 2134)$ is the set of parking functions of length n whose parking permutations avoid 1234 and 2134.
- (3) $\text{pk}_n(1234, 2134)$ is the number of parking functions in the set $\text{Pk}_n(1234, 2134)$, i.e.

$$\text{pk}_n(1234, 2134) = |\text{Pk}_n(1234, 2134)|.$$

For brevity, we let $\text{pk}(n) = \text{pk}_n(1234, 2134)$. For each bi-suffix $(i, r) \in B_n^{(1)}(1234, 2134)$, let

- (1) $\text{pk}(n, i)$ denote the number of parking functions whose parking permutations π satisfy $\pi_n = i$,
- (2) $\text{pk}(n, i, r)$ denote the number of parking functions whose parking permutations π have bi-suffix (i, r) .

Claim 3. For $1 \leq i \leq n$,

$$\text{pk}(n, i) = \sum_{r=1}^i \text{pk}(n, i, r).$$

Take $\pi \in \text{Av}_n(1234, 2134)$ and suppose that $\pi_n = i$. If

$$r = \ell(\pi_n = i) = \max\{r : \pi_j \leq \pi_n = i \text{ for all } n-r+1 \leq j \leq n\},$$

then $1 \leq r \leq i$, as there are $i-1$ possible elements contiguously to the left of $\pi_n = i$ smaller than $\pi_n = i$.

Theorem 4.4. Let $\text{pk}(n) = \text{pk}_n(1234, 2134)$ be the number of parking functions whose parking permutations avoid the patterns 1234 and 2134. Then

$$\text{pk}(n) = \sum_{i=1}^n \text{pk}(n, i),$$

with $\text{pk}(0) = 1$. For each bi-suffix $S = (i, r) \in B_n^{(1)}(1234, 2134)$, let $\text{pk}(n, i, r)$ denote the number of parking functions whose parking permutations π have bi-suffix S . Then

$$\text{pk}(n, i) = \sum_{r=1}^i \text{pk}(n, i, r),$$

and

$$\text{pk}(n, i, r) = \begin{cases} \sum_{j=i}^{n-1} \text{pk}(n-1, j) & \text{if } r = 1 \\ 2 \text{pk}(n-2) & \text{if } r = 2 \text{ and } i = 2 \\ 4 \text{pk}(n-1, i-1, 1) & \text{if } r = 2 \text{ and } i \neq 2 \\ \text{pk}(n-r) \text{pk}(r, r) & \text{if } i = r \text{ and } i < n \\ \frac{n}{n-1} \text{pk}(n-1, n-1, n-1) + \sum_{r_1=2}^{n-1} \frac{2n}{n-r_1} \text{pk}(n-r_1, n-r_1, n-r_1) & \text{if } i = r \text{ and } i = n \\ \frac{r}{r-1} \text{pk}(n-1, i-1, r-1) + r \text{pk}(n-r+1, i-r+1, 1) & \text{otherwise} \\ + \sum_{r_1=2}^{r-1} \frac{2r}{r-r_1} \text{pk}(n-r_1, i-r_1, r-r_1) & \end{cases}$$

with

$$\text{pk}(1, i, r) = \begin{cases} 1, & (i, r) = (1, 1) \\ 0, & \text{otherwise,} \end{cases} \quad \text{pk}(2, i, r) = \begin{cases} 1, & (i, r) = (1, 1) \\ 2, & (i, r) = (2, 2) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\text{pk}(3, i, r) = \begin{cases} 3, & (i, r) = (1, 1) \\ 2, & (i, r) = (2, 1) \text{ or } (i, r) = (2, 2) \\ 9, & (i, r) = (3, 3) \\ 0, & \text{otherwise.} \end{cases}$$

The first terms of the sequence are

$$1, 3, 16, 89, 534, 3394, 22447, 152901, 1065508, 7560824, \dots$$

The strategy of the proof is as follows. We begin with a permutation π of length n and parking table p . By the permutation scheme, we reduce the permutation π by deleting elements according to the scheme. This deletion causes some elements of the parking table p to change (depending on the deletion). Since the product of the elements of p yields the number of parking functions that lead to the parking permutation π , to recover this number, we multiply the new product of the elements in the new parking table by some factor that depends on the deletion.

Remark. Whenever a word w of length n is reduced to a permutation $\text{red}(w)$ of length m , we simply write $\text{red}(w) = w_1 w_2 \cdots w_m$ where for each k , w_k is the corresponding element after reduction. For the main proof, we only keep track of the last two elements depending on deletion.

Proof. Let $\Pi = \{1234, 2134\}$ and $\pi \in \text{Av}_n(\Pi)$ such that $\pi_{n-1} = j, \pi_n = i$ and $p = (p_1, p_2, \dots, p_n)$ be the parking table of π such that $p_n = r$. We prove the recurrence via the following cases on the value of r :

- (1) If $r = 1$: In this case, $\pi_{n-1} > \pi_n$ and so π_{n-1}, π_n reduce to $2, 1$. Hence, $i + 1 \leq j \leq n$. By the permutation scheme, delete $\pi_n = i$ from π and apply the reduction map to obtain a permutation of length $n - 1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-2} j$ and reducing w to a permutation of length $n - 1$ results in a permutation whose last element is $j - 1$. Thus,

$$\text{pk}(n, i, r) = \sum_{j=i}^{n-1} \text{pk}(n-1, j).$$

- (2) If $r = 2$: In this case, if $r = 2$ then $j = \pi_{n-1} < \pi_n = i$ and $\pi_n < \pi_{n-2}$. Hence, we proceed with the cases $\pi_n = 2$ or $\pi_n \neq 2$ from the permutation scheme.

- (a) If $\pi_n = 2$ then $\pi_{n-1} = 1$ and the parking table is $p = (p_1, p_2, \dots, 1, 2)$. By the permutation scheme, delete $\pi_{n-1} = 1$ from π and apply the reduction map to obtain a permutation of length $n - 1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-2} 2$ and reducing w to a permutation of length $n - 1$ results in $\text{red}(w) = w_1 w_2 \cdots w_{n-2} 1$. The parking table of w and $\text{red}(w)$ coincide for the first $n - 2$ elements by Claim 2. The corresponding parking table of $\text{red}(w)$ is $p' = (p_1, p_2, \dots, p_{n-2}, 1)$ since the last element of $\text{red}(w)$ equals to one. Furthermore, since $\text{red}(w)$ ends with the element one, deleting it and reducing again yields a permutation of length $n - 2$ and its parking table is $p'' = (p_1, p_2, \dots, p_{n-2})$. In particular, to recover the product $p_1 p_2 \cdots p_{n-2} 12$, we multiply the product $p_1 p_2 \cdots p_{n-2}$ by 2, which results in the number of parking functions whose parking permutations avoid Π . Thus, $\text{pk}(n, i, r) = 2 \text{pk}(n - 2)$.

- (b) If $\pi_n \neq 2$ with $j < i$ and $r = 2$ then $j = 1$ or $j = 2$ by the permutation scheme. Since $r = 2$ then $\pi_{n-2} > \pi_n = i$.

- (i) If $j = 1$ then $\pi = \pi_1 \pi_2 \cdots 1 i$ ($i > 2$) and the parking table is $p = (p_1, p_2, \dots, 1, 2)$. By the permutation scheme, delete $\pi_{n-1} = 1$ and apply the reduction map to obtain a permutation of length $n - 1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-2} i$ and reducing w to a permutation of length $n - 1$ results in a permutation whose last element is $i - 1$ (since $j = 1 < i$). Thus, $\text{red}(w) = w_1 w_2 \cdots w_{n-2} (i - 1)$. The corresponding parking table of $\text{red}(w)$ is $p' = (p_1, p_2, \dots, p_{n-2}, 1)$ since $w_{n-2} \geq i > i - 1$ as $\pi_{n-2} > \pi_n = i$. Moreover, to recover the product $p_1 p_2 \cdots p_{n-2} 12$, we multiply the product $p_1 p_2 \cdots p_{n-2}$ by 2, which results in the number $\text{pk}(n, i, r)$. Thus, $\text{pk}(n, i, r) = 2 \text{pk}(n - 1, i - 1, 1)$.

- (ii) If $j = 2$ then $\pi = \pi_1 \pi_2 \cdots 2 i$ ($i > 2$) and the parking table is $p = (p_1, p_2, \dots, 1, 2)$ (since $\pi_{n-2} > i > 2$). Then again, delete $\pi_{n-1} = 2$ and a similar argument as in the case for $j = 1$ shows that $\text{pk}(n, i \neq 2, r) = 2 \text{pk}(n - 1, i - 1, 1)$.

Altogether, for case (2b), we obtain $\text{pk}(n, i \neq 2, r) = 2 \text{pk}(n-1, i-1, 1) + 2 \text{pk}(n-1, i-1, 1) = 4 \text{pk}(n-1, i-1, 1)$.

The next two cases are broken up into the cases $r = i < n$ or $r = i = n$.

- (3) If $2 < r < n$ and $r = i$: In this case, $r > 2$ implies that $j = \pi_{n-1} < \pi_n = i$. By the permutation scheme $j = 1$ or $j = 2$. Moreover, π can be decomposed in the following form:

$$\pi = \underbrace{\pi_1 \pi_2 \cdots \pi_{n-i}}_{n-i} \underbrace{\pi_{n-i+1} \pi_{n-i+2} \cdots \pi_{n-2}}_{i-1} j^i$$

where $\pi_k < \pi_n = i$ for any $k, n-i < k < n$, and $\pi_{n-i} > \pi_n = i$. In particular, if $j = 1$ then

$$\{\pi_{n-i+1}, \pi_{n-i+2}, \dots, \pi_{n-2}\} = \{2, 3, \dots, i-2, i-1\},$$

and if $j = 2$ then

$$\{\pi_{n-i+1}, \pi_{n-i+2}, \dots, \pi_{n-2}\} = \{1, 3, \dots, i-2, i-1\}.$$

This is true because every element in the set $\{\pi_{n-i+1}, \pi_{n-i+2}, \dots, \pi_{n-2}, j\}$ is smaller than i and there are $i-1$ elements in the set. Thus, the number of parking functions whose parking permutations avoid the patterns Π is equal to $\text{pk}(n, i, r) = \text{pk}(n-i) \text{pk}(i, i)$ with $i = r$. Hence, $\text{pk}(n, i, r) = \text{pk}(n-r) \text{pk}(r, r)$.

- (4) If $r = n$ and $r = i$: In this case, $r = n$ implies that $j = \pi_{n-1} < \pi_n = n$. By the permutation scheme $j = 1$ or $j = 2$. The key to this case is noting that decomposing π into two block as in case (3) is not possible. Indeed, all elements from π_1 to π_{n-1} are less than $\pi_n = n$. Hence, we proceed with the cases $\pi_{n-1} = 1$ or $\pi_{n-1} = 2$ from the permutation scheme.

- (a) If $\pi_{n-1} = 1$ then $\pi = \pi_1 \pi_2 \cdots 1n$ and the parking table is $p = (p_1, p_2, \dots, 1, n)$. By the permutation scheme, delete $\pi_{n-1} = 1$ and apply the reduction map to obtain a permutation of length $n-1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-2}n$ and reducing w to a permutation of length $n-1$ results in $\text{red}(w) = (\pi_1 - 1)(\pi_2 - 1) \cdots (\pi_{n-2} - 1)(n-1)$, since for every $k \in [n-2]$, we have $\pi_k > \pi_{n-1} = 1$. The corresponding parking table of $\text{red}(w)$ is $p' = (p_1, p_2, \dots, p_{n-2}, n-1)$, where the first $n-2$ elements coincide with the parking table of p by Claim 2. Therefore, to recover the product $p_1 p_2 \cdots p_n$, we multiply $p_1 p_2 \cdots p_{n-2}(n-1)$ by $\frac{n}{n-1}$. Hence $\text{pk}(n, i, r) = \frac{n}{n-1} \text{pk}(n-1, n-1, n-1)$.
- (b) If $\pi_{n-1} = 2$ then $\pi = \pi_1 \pi_2 \cdots 2n$ and the parking table is $p = (p_1, p_2, \dots, p_{n-1}, n)$. Then there exists an index k such that $\pi_k = 1$ for some $1 \leq k \leq n-2$. Let k be such index and consider the subsequence $w = 1\pi_{k+1}\pi_{k+2} \cdots 2$ of π . By Lemma 4.2 together with $\pi_{k+1} > \pi_k = 1$, we obtain that the parking table of w is $p' = (1, 2, 1, \dots, 1)$ since $\pi_a > \pi_b$ for any $a < b$. Suppose that $w = 1\pi_{k+1}\pi_{k+2} \cdots 2$ is of length r_1 , where $2 \leq r_1 \leq n-1$. The bounds on r_1 follows from observing that w can be at most length $n-1$ if $\pi_1 = 1$ and it can be at least length 2 if $\pi_{n-2} = 1$. Hence, the parking table p of π is equal to

$$p = (p_1, p_2, \dots, p_{r_1-1}, 1, 2, \underbrace{1, \dots, 1}_{r_1-2}, n).$$

Delete $w = 1\pi_{k+1}\pi_{k+2} \cdots 2$ from π and apply the reduction map to obtain a permutation of length $n-r_1$. The resulting word after deletion is $w' = \pi_1 \pi_2 \cdots \pi_{n-r_1-1}n$ and reducing w' to a permutation of length $n-r_1$ results in $\text{red}(w') = w'_1 w'_2 \cdots (n-r_1)$. The corresponding parking table of $\text{red}(w')$ is $q = (p_1, p_2, \dots, p_{n-r_1-1}, n-r_1)$. This follows by the deletion of exactly r_1 elements all less than $\pi_n = n$. Therefore, to recover the product

$$p_1 p_2 \cdots p_{n-r_1-1} 1 2 \underbrace{1 \cdots 1}_{r_1-2} n = 2n p_1 p_2 \cdots p_{n-r_1-1},$$

we multiply $p_1 p_2 \cdots p_{n-r_1-1}(n-r_1)$ by $\frac{2n}{n-r_1}$ for all $2 \leq r_1 \leq n-1$. Hence $\text{pk}(n, i, r) = \sum_{r_1=2}^{n-1} \frac{2n}{n-r_1} \text{pk}(n-r_1, n-r_1, n-r_1)$.

Altogether, for case (4), we obtain

$$\text{pk}(n, i, r) = \frac{n}{n-1} \text{pk}(n-1, n-1, n-1) + \sum_{r_1=2}^{n-1} \frac{2n}{n-r_1} \text{pk}(n-r_1, n-r_1, n-r_1).$$

- (5) The last case is if $r \neq i$. In this case, if $r < i$ then $i < n$ because if not, $i = n$ implies $r = i$, which is the previous case. Thus $2 < r < i$ and $j = \pi_{n-1} < \pi_n = i$. Hence, we proceed with the two cases $\pi_{n-1} = 1$ or $\pi_{n-1} = 2$ from the permutation scheme.

- (a) If $\pi_{n-1} = 1$ then $\pi = \pi_1 \pi_2 \cdots 1i$ and the parking table is $p = (p_1, p_2, \dots, 1, r)$. By the permutation scheme, delete $\pi_{n-1} = 1$ and apply the reduction map to obtain a permutation of length $n-1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-2}i$ and reducing w to a permutation of length $n-1$ results in $\text{red}(w) = w_1 w_2 \cdots w_{n-2}(i-1)$. The corresponding parking table of $\text{red}(w)$ is $p' = (p_1, p_2, \dots, p_{n-2}, r-1)$. Hence, to recover the product $p_1 p_2 \cdots p_{n-2} 1r = r p_1 p_2 \cdots p_{n-2}$, we multiply $p_1 p_2 \cdots p_{n-2}(r-1)$ by $\frac{r}{r-1}$. It follows that $\text{pk}(n, i, r) = \frac{r}{r-1} \text{pk}(n-1, i-1, r-1)$.
- (b) If $\pi_{n-1} = 2$ then $\pi = \pi_1 \pi_2 \cdots 2i$ and the parking table is $p = (p_1, p_2, \dots, p_{n-1}, r)$. This implies that $\pi_n = i > \pi_k$ for every $n-r+1 \leq k \leq n-1$ and $\pi_{n-r} > \pi_n = i$. Hence, either $\pi_k = 1$ for some $1 \leq k \leq n-r-1$ or $\pi_k = 1$ for some $n-r+1 \leq k \leq n-2$. Observe that $\pi_{n-r} \neq 1$ as this would imply that $1 = \pi_{n-r} < \pi_n = i$, which is not the case. We proceed with the two cases based on the position of the element 1.

- (i) If $\pi_k = 1$ for some $k \in \mathbb{N}$, $1 \leq k \leq n-r-1$ then since $r > 2$, $\pi_a > \pi_b$ for all a, b such that $n-r+1 \leq a < b \leq n-2$. Otherwise if $\pi_a < \pi_b$ then $\pi_k = 1, \pi_a, \pi_b, \pi_n = i$ is order isomorphic to 1234, a contradiction. Hence, $\pi_a > \pi_b$ for all a, b with $n-r+1 \leq a < b \leq n-2$. It follows that $\pi_{n-r+1} > \pi_{n-r+2} > \cdots > \pi_{n-2}$. Thus, the parking table of the word

$$(\pi_{n-r+1}, \pi_{n-r+2}, \dots, \pi_{n-2}, 2)$$

is equal to

$$(1, 1, \dots, 1, 1),$$

with $r-1$ ones. Therefore, the parking table of

$$\pi = \pi_1 \pi_2 \cdots \pi_{n-r} \pi_{n-r+1} \cdots \pi_{n-2} 2i$$

is

$$p = (p_1, p_2, \dots, p_k = 1, p_{k+1}, \dots, p_{n-r}, \underbrace{1, 1, \dots, 1}_{r-1}, r).$$

Delete $\pi_{n-r+1} \pi_{n-r+2} \cdots \pi_{n-2} 2$ from π and apply the reduction map to obtain a permutation of length $n-r+1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-r} i$ and reducing w to a permutation of length $n-r+1$ results in $\text{red}(w) = w_1 w_2 \cdots (i-r+1)$. This follows by the deletion of exactly $r-1$ elements smaller than $\pi_n = i$. The corresponding parking table of $\text{red}(w)$ is

$$p' = (p_1, p_2, \dots, p_{n-r}, 1),$$

since $\pi_n = i < \pi_{n-r}$ and hence $w_{n-r+1} = i-r+1 < w_{n-r} = \pi_{n-r} - r + 1$ as exactly $r-1$ elements are deleted all less than π_{n-r} . Hence, to recover the product

$$p_1 p_2 \cdots p_{n-r} \underbrace{1 \cdots 1}_{r-1} r = r p_1 p_2 \cdots p_{n-r},$$

we multiply $p_1 p_2 \cdots p_{n-r} 1$ by r . It follows that $\text{pk}(n, i, r) = r \text{pk}(n-r+1, i-r+1, 1)$.

- (ii) If $\pi_k = 1$ for some $k \in \mathbb{N}$, $n-r+1 \leq k \leq n-2$, then

$$\pi = \pi_1 \pi_2 \cdots \pi_{n-r} \pi_{n-r+1} \cdots \pi_{k-1} 1 \pi_{k+1} \cdots \pi_{n-2} 2i$$

and the parking table is

$$p = (p_1, p_2, \dots, p_{n-r}, p_{n-r+1}, \dots, p_{k-1}, 1, p_{k+1}, \dots, p_{n-2}, 2, r).$$

Observe that $\pi_k = 1 < \pi_{k+1}$ and $\pi_a > \pi_b$ for all a, b such that $k+1 \leq a < b \leq n-1$. Otherwise, if $\pi_a < \pi_b$ then $\pi_k = 1, \pi_a, \pi_b, \pi_n = i$ is order isomorphic to 1234, a contradiction. Hence, $\pi_k = 1 < \pi_{k+1}$ and $\pi_a > \pi_b$ for all a, b such that $k+1 \leq a < b \leq n-1$. It follows that $\pi_{k+1} > \pi_{k+2} > \cdots > \pi_{n-2} > 2$. Let $1 \pi_{k+1} \pi_{k+2} \cdots \pi_{n-2} 2$ be of length r_1 for $2 \leq r_1 \leq r-1$. The bounds on r_1 follows from observing that $r_1 = 2$ if

$\pi_{n-2} = 1$, and at most equal to $r - 1$ if $\pi_{n-r+1} = 1$ and $\pi_{n-r} \neq 1$. Hence, the parking table of the word

$$(\pi_k = 1, \pi_{k+1}, \dots, \pi_{n-2}, 2)$$

is equal to

$$(1, 2, \underbrace{1, \dots, 1}_{r_1-2}).$$

Thus, the parking table of

$$\pi = \pi_1 \pi_2 \cdots \pi_{n-r} \pi_{n-r+1} \cdots \pi_{k-1} 1 \pi_{k+1} \cdots \pi_{n-2} 2 i,$$

is

$$p = (p_1, p_2, \dots, p_{n-r}, p_{n-r+1}, \dots, p_{n-r_1-2}, p_{n-r_1-1}, 1, 2, \underbrace{1, \dots, 1}_{r_1-2}, r).$$

Delete $1 \pi_{k+1} \pi_{k+2} \cdots \pi_{n-2} 2$ from π and apply the reduction map to obtain a permutation of length $n - r_1$. The resulting word after deletion is $w = \pi_1 \pi_2 \cdots \pi_{n-r} \pi_{n-r+1} \cdots \pi_{n-r_1-1} i$ and reducing w to a permutation of length $n - r_1$ results in

$$\text{red}(w) = w_1 w_2 \cdots w_{n-r} r_{n-r+1} \cdots w_{n-r_1-1} (i - r_1).$$

The corresponding parking table of $\text{red}(w)$ is

$$p' = (p_1, p_2, \dots, p_{n-r}, p_{n-r+1}, \dots, p_{n-r_1-1}, r - r_1).$$

This follows by the deletion of exactly r_1 elements smaller than $\pi_n = i$. Thus, to recover the product

$$p_1 p_2 \cdots p_{n-r} p_{n-r+1} \cdots p_{n-r_1-2} p_{n-r_1-1} \underbrace{1 \cdots 1}_{r_1-2} r = 2r p_1 p_2 \cdots p_{n-r_1-2} p_{n-r_1-1},$$

we multiply $p_1 p_2 \cdots p_{n-r} p_{n-r+1} \cdots p_{n-r_1-1} (r - r_1)$ by $\frac{2r}{r-r_1}$. Thus, for all $2 \leq r_1 \leq r - 1$,

$$\text{pk}(n, i, r) = \sum_{r_1=2}^{r-1} \frac{2r}{r-r_1} \text{pk}(n-r_1, i-r_1, r-r_1).$$

Altogether, for case (5), we obtain

$$\begin{aligned} \text{pk}(n, i, r) &= \frac{r}{r-1} \text{pk}(n-1, i-1, r-1) + r \text{pk}(n-r+1, i-r+1, 1) \\ &\quad + \sum_{r_1=2}^{r-1} \frac{2r}{r-r_1} \text{pk}(n-r_1, i-r_1, r-r_1). \end{aligned}$$

In all of the arguments above, by Claim 1, each reduced permutation avoids both patterns 1234, 2134. In addition, it is easy to check that the initial conditions are satisfied for $n \leq 3$. $\square$

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