A Maple Rendition of George Polya's Elementary Proof[α] of the Modularity of the Theta Function

Dr. Z.'s Experimental Math Spring 2018 class [β]

Trivial Fact 1[γ]**:**

$$(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^{2m} = \sum_{v=-m}^{m} \binom{2m}{m+v} z^v \quad .$$

Trivial Fact 2[δ] : For any (Laurent) polynomial $P(z) = \sum_{N_1 < v < N_2} a_v\, z^v$ and any positive odd integer l, if $\omega = e^{2\pi i/l}$,

$$\sum_{-\frac{l}{2} < v < \frac{l}{2}} P(\omega^v z) \, = \, l \sum_{\frac{N_1}{l} < v < \frac{N_2}{l}} a_{lv} z^{lv} \quad .$$

Combining (and dividing by 2^{2m}):

$$\sum_{-\frac{l}{2} < v < \frac{l}{2}} \frac{1}{2^{2m}} \left((\omega^v\, z)^{1/2} + (\omega^v\, z)^{-1/2} \right)^{2\,m} \, = \, l \sum_{-\frac{m}{l} < v < \frac{m}{l}} \frac{1}{2^{2m}} \binom{2m}{m + l\,v} z^{lv} \quad .$$

Now let $z = e^{s/l}$ and $l = \lfloor \sqrt{m\,t} \rfloor$, and take the limits as $m \to \infty$. Go to `maple`, and *copy-and-paste*:

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l:=sqrt(m*t): z:=exp(s/l): omega:=exp(2*Pi*I/l): L:= ( ( ((omega**v*z)**(1/2) +
(omega**v*z)**(-1/2)))/2)**(2*m): R:= l*binomial(2*m,m+l*v)*z**(l*v)/2**(2*m):
factor(limit(L, m=infinity)); simplify(limit(R,m=infinity));
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and get, respectively,

$$\mathrm{e}^{\frac{-(2\,\pi\,v - i s)^2}{4t}} \quad , \quad \frac{\sqrt{t}\,\mathrm{e}^{v(-tv+s)}}{\sqrt{\pi}} \quad ,$$

yielding:

$$\sum_{v=-\infty}^{\infty} e^{(s+2\pi i v)^2/(4t)} \, = \, \left(\frac{t}{\pi}\right)^{1/2} \sum_{v=-\infty}^{\infty} e^{-tv^2 + sv} \quad . \quad \square$$

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[α] G. Polya, Sitz. der Phys.-Math. Klasse, Berlin (1927), 158-161. See also R. Bellman, "A Brief Introduction to Theta Functions" (1961), pp. 40-41.

[β] Yonah Biers-Ariel, Matthew P. Charnley, Amit Harel, George D. Hauser, Edna L. Jones, Ahsan Z. Khan, Brooke Logan, Yukun Yao, Anthony Zaleski. Instructor: Doron Zeilberger.

[γ] Binomial Theorem.

[δ] Sum of a geometric series $\sum_{-l/2 < v < l/2} z^v = \frac{z^{l/2} - z^{-l/2}}{z^{1/2} - z^{-1/2}}$.