

von Neumann and Newman Pokers with Finite Decks

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Abstract

John von Neumann studied a simplified version of poker where the “deck” consists of *infinitely* many cards, in fact, all real numbers between 0 and 1. We harness the power of computation, both numeric and symbolic, to investigate analogs with *finitely* many cards. We also study finite analogs of a simplified poker introduced by D.J. Newman, and conclude with a thorough investigation, fully implemented in Maple, of the three-player game, doing *both* the finite and the infinite versions. This paper is accompanied by two Maple packages and numerous output files; however, no knowledge of Maple is needed, as all relevant information is provided within the paper.

1 Prelude

Welcome to the world of poker, where strategy and probability rule. Picture yourself at the poker table, every decision a crucial step toward victory or defeat. Poker has intrigued mathematicians for decades as a window into decision-making and game theory. Pioneers like Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley ([1],[5],[3],[4]) who believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking, have simplified the complexities of the game, making it tractable for game theoretic analysis.

Quick Refresher: Game Theory

In game theory, a *game* refers to any situation where players make decisions that result in outcomes based on the choices of all involved. A *strategy* is a complete plan of action a player will follow in various situations throughout the game. A *pure strategy* is a strategy in which a player makes a specific choice or takes a specific action with certainty whenever a particular situation arises in the game. When players use a *mixed strategy*, they randomize over possible moves, assigning a probability to each move, instead of choosing a single, deterministic action. A *Nash Equilibrium (NE)* occurs when no player can benefit from changing their strategy while the other players keep theirs unchanged. A *mixed Nash Equilibrium* is a type of NE where at least one player uses a mixed strategy, ensuring no player can improve their payoff by individually changing their strategy.

von Neumann Poker

In the original version [5], von Neumann proposed, and solved, the following game of poker with an uncountably infinite deck, namely all the real numbers between 0 and 1. Fix a bet size, b . Player I and Player II are dealt (uniformly at random) two “cards”,

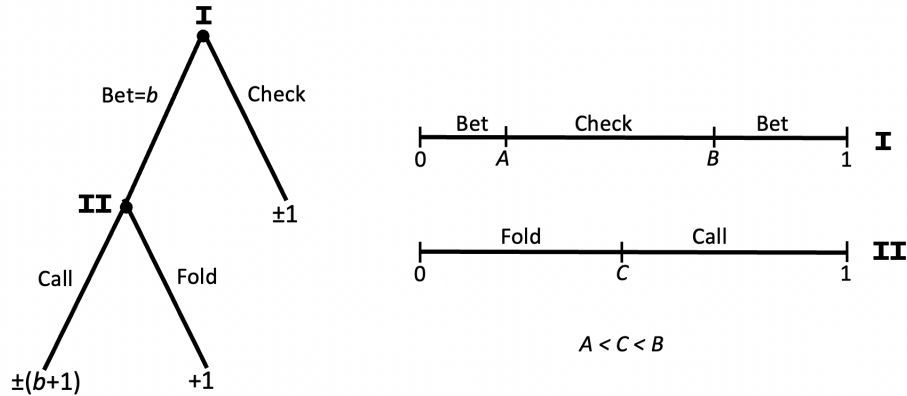


Figure 1: The betting tree and Nash equilibrium strategies for von Neumann Poker

real numbers x and y , in the interval $[0, 1]$. They each see their own card, but have no clue about the opponent's card. At the start they each put one dollar into the pot (the so called *ante*), so now the pot has two dollars.

Figure 1 illustrates the “betting tree” of this game. Here, Player I looks at his card, and decides whether to *check*, in which case each of the players shows their cards, and whoever has the largest card wins the pot. On the other hand he has an option to *bet*, putting b additional dollars in the pot. Now the game turns to Player II. She can decide to *fold*, in which case player I gets the pot, resulting in a gain of 1 dollar for Player I, (and a loss of 1 dollar for player II), or be **brave** and *call*, putting her own b dollars into the pot, that now has $2b + 2$ dollars. The cards are compared in a *showdown* and whoever has the larger card, wins the whole pot, resulting in a gain of $b + 1$ dollars for the winner, and a loss of $b + 1$ for the loser.

von Neumann proved that the following pair of strategies is a pure *Nash Equilibrium*, i.e. if the players both follow their chosen strategy, neither of them can do better (on average) by doing a different strategy.

The von Neumann advice

von Neumann identified the cuts A, B and C in the right panel of Figure 1, and proposed the following strategies.

- Player I: If $0 < x < \frac{b}{(b+4)(b+1)}$ or $\frac{b^2 + 4b + 2}{(b+4)(b+1)} < x < 1$ you should **bet**, otherwise **check**.
- Player II: If $0 < y < \frac{b(b+3)}{(b+4)(b+1)}$ you should **fold**, otherwise **call**.

Note that Player II's strategy corresponds to *honest common sense*, there is some *cut-off* that below it you should be conservative, and “cut your losses” giving up the one dollar, and not risking losing b additional dollars, and above it, be brave, and *go for it*.

Now an *honest common sense* would tell you that Player I would also have his own cutoff, check if his card is below it, and bet if it exceeds it. But this is **not** optimal. If Player I has a low card, he should **bluff**, and ‘pretend’ that he has a high card, and player II would be intimidated into folding.

Sad but true, “honesty is **not** the best policy”. Indeed the game favors Player I, and his expected gain is $\frac{b}{(b+4)(b+1)}$.

When $b = 2$, the advice spells out as follows:

- Player I: if $0 < x < \frac{1}{9}$ or $\frac{7}{9} < x < 1$ you should **bet**, otherwise **check**.
- Player II: If $0 < y < \frac{5}{9}$ you should **fold**, otherwise **call**.

The expected value, i.e. the value of the game (for Player I) is $\frac{1}{9}$. It can be shown that $b = 2$ maximizes Player I’s payoff under the Nash equilibrium strategies.

Finitely Many Cards

What we *don’t* like about the original von Neumann version is that the deck is infinite. In **real life** there are only *finitely* many cards, and in fact, not that many. We were wondering whether there exists pure Nash equilibria when there are only finitely many cards.

For interested readers, we hope you will download the Maple package `FinitePoker.txt`, available, free of charge, at <https://sites.math.rutgers.edu/~zeilberg/tokhniot/FinitePoker.txt>. Once you downloaded our Maple package to your laptop, that has Maple, and set the directory to be the one where the package resides, start a worksheet and type `read `FinitePoker.txt``.

We wrote procedure `vnNE(n, b)`, that inputs

- a positive integer n , at least 2, standing for the number of cards in the deck, that are numbered $1, 2, \dots, n$.
- a positive integer b , at least 1, denoting the (fixed) *bet size*.

It outputs the set of **all** pure Nash equilibria. This set may be empty, since we are talking about **pure** NEs (from now on NE:=Nash Equilibrium). Recall that thanks to John Nash, we are only **guaranteed** the existence of *mixed* NEs. We will talk about mixed NEs later on. But for now, let’s explore the pure ones.

Finding all pure Nash Equilibria via the “Vanilla” approach

We did not make *any* assumptions about ‘plausible’ strategies, so *a priori*, a strategy for player I can be *any* subset, S_1 , of $\{1, \dots, n\}$, that advises: ‘If your card belongs to S_1 you should **bet**, otherwise, **check**’. Similarly a strategy for player II, S_2 , can be any such subset, that tells her to call iff her card $j \in S_2$. For each conceivable strategy pair $[S_1, S_2]$ we can easily compute the expected payoff following these strategies. This is implemented in procedure `EnS1S2(n, S1, S2, b)`.

Using this, we can construct the *paytable*, implemented in procedure `PayTable(n, b)`, that is a 2^n by 2^n matrix. Now we look for pure NEs, the usual way, by finding, for *each* strategy of each player the *best response* of the other player, and looking for pairs $[S_1, S_2]$ that are best responses to each other.

Let’s fix the bet size $b = 2$. If the card has only 2 cards, `vnNE(2, 2)`; gives

$$\{ \{ \{ \}, \{ 2 \}, 0 \}, \{ \{ 2 \}, \{ 2 \}, 0 \} \} \quad ,$$

so there are two pure NEs. In both of them Player II bets if her card is 2 and folds if her card is 1, while Player I always checks in the first strategy, and checks if his card is 1 in the second strategy.

This is not very interesting, since the expected gain (value of the the game) is 0.

$\text{vnNE}(3, 2)$ is equally boring, giving the two trivial pairs $[\phi, \{3}]$ and $[\{3\}, \{3}]$.

$\text{vnNE}(4, 2)$, $\text{vnNE}(5, 2)$, and $\text{vnNE}(6, 2)$ are even more boring, they are empty!

But now comes a nice surprise, $\text{vnNE}(7, 2)$ gives three pure, *non-trivial*, NEs.

For all of them Player I bets iff his card belongs to $\{1, 6, 7\}$, but Player II calls if her card is in either $\{3, 6, 7\}$, $\{4, 6, 7\}$, or $\{5, 6, 7\}$. The value of the game is $\frac{2}{21}$.

So with 7 cards we already have bluffing! If Player I has the card labeled 1, he should bet even though he would definitely lose the bet if Player II calls.

Moving right along, $\text{vnNE}(8, 2)$; also gives you three pure NEs.

For all of them Player I bets iff his card belongs to $\{1, 7, 8\}$, but Player II calls if her card is in either $\{4, 7, 8\}$, $\{5, 7, 8\}$, or $\{6, 7, 8\}$. The value of the game is $\frac{3}{28}$, getting tantalizingly close to von Neumann's $\frac{1}{9}$.

Since the sizes of the payoff matrices grow exponentially, and we did not make *any plausibility assumptions*, there is only so far we can go with this naive *vanilla* approach. But nine cards are still doable. Indeed there are seven pure NEs in this case. For all of them $S_1 = \{1, 8, 9\}$, but Player II has seven choices, all with four members, including, of course, $\{6, 7, 8, 9\}$.

For all pure NEs for n from 2 to 10 and bet-sizes from 1 to 5 look at the output file: <https://sites.math.rutgers.edu/~zeilberg/tokhniot/oFinitePoker1.txt>.

To overcome the *exponential explosion*, we can stipulate that Player I's strategy **must** be of the form:

“Check iff $i \in \{A, A + 1, \dots, B\}$ for some $1 \leq A < B \leq n$, ”

while Player II's must be of the form:

“Call iff $j \in \{C, C + 1, \dots, n\}$ for some $1 \leq C \leq n$.”

Now we can go much further, see the output file at

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/oFinitePoker1A.txt> for pure NEs for n up to 27.

If n is a multiple of 9 then the (restricted) pure NEs are as expected, namely the value of the game is $\frac{1}{9}$ and the strategy for player I is: check if $\frac{1}{9}n < i \leq \frac{7}{9}n$, bet otherwise and for Player II: call iff $j > \frac{5}{9}n$.

If n is not a multiple of 9, then the values are close, but a little less. For example for $n = 26$ the value is $\frac{36}{325} = 0.110769$. For $n = 25$ the value is $\frac{11}{100} = 0.11$, for $n = 24$ it is $\frac{61}{552} = 0.1105072464$, for $n = 23$ it is $\frac{28}{253} = 0.1106719368$, for $n = 22$ it is $\frac{17}{154} = 0.1103896104$.

2 Mixed NEs via Linear Programming

The study of mixed strategies in two-person zero-sum games can be elegantly formulated as a primal-dual linear programming (LP) problem. A *mixed* strategy involves each player choosing optimal actions according to a probability distribution, introducing uncertainty. An equilibrium solution to this dual pair of linear programs reveals optimal mixed strategies (mixed NE) for both players.

Slow LPs for mixed NE

Recall our scenario: the pot starts at 1+1, with only Player I able to bet a fixed amount b . Given the 2^n by 2^n payoff matrix (m_{ij}) as input, Player I aims to maximize his worst-case expected gain, minimizing over all possible actions of Player II. This objective is framed as an LP by introducing variable v_1 to represent this minimum, ensuring Player I's expected gain is at least v_1 for every action of Player II, and maximizing v_1 . Similarly, from Player II's viewpoint, the goal is to minimize her worst-case expected loss, maximizing over all actions of Player I. This involves introducing variable v_2 to represent this maximum, and setting the objective to minimize v_2 .

To formulate the primal-dual LP, let $\mathbf{x} = (x_1, \dots, x_{2^n})$ be the mixed strategy probability of Player I to maximize v_1 . Let $\mathbf{y} = (y_1, \dots, y_{2^n})$ be the mixed strategy probability of Player II to minimize v_2 .

$$\begin{array}{ll}
 \text{Primal: Maximize } v_1 & \text{Dual: Minimize } v_2 \\
 \text{s.t. } \sum_{i=1}^{2^n} x_i \cdot m_{ij} \geq v_1 \quad \text{for } j = 1, \dots, 2^n & \text{s.t. } \sum_{j=1}^{2^n} m_{ij} \cdot y_j \leq v_2 \quad \text{for } i = 1, \dots, 2^n \\
 \sum_{i=1}^{2^n} x_i = 1 & \sum_{j=1}^{2^n} y_j = 1 \\
 x_i \geq 0 \quad \text{for } i = 1, \dots, 2^n. & y_j \geq 0 \quad \text{for } j = 1, \dots, 2^n.
 \end{array}$$

By the minimax theorem at an equilibrium, $v_1 = v_2 = v^*$, which represents the value of the game. We can use the commands `maximize` and `minimize` in the Maple package `simplex`, to find **one** mixed NE. However, due to the exponentially large size of the matrix, practical limitations arise, restricting us from considering more than 6-7 cards without the inconvenience of reducing the dominated rows/columns of the payoff matrix (worse than the vanilla approach in the previous section).

We have implemented the above Slow LP procedures in `MNE(M, S1, S2)`. Typing

```

M := PayTable(4, 2) [1]:
S1 := Stral(4):
S2 := Stra2(4):
MNE(M, S1, S2);    gives outputs:
[{{[4], 1/2}, [{1, 4], 1/2}}, {[4], 1/2}, [{2, 4], 1/2}], 1/12] .

```

Translation:

- The value of the game is $\frac{1}{12}$ (last entry).
- Player I has two strategies specified within the first set of braces: (1.1) with probability 1/2, raise if his card is 4 and fold if his cards are 1, 2, or 3; and (1.2) with probability 1/2, raise if his card is 1 or 4 and fold if his cards are 2 or 3.
- Player II has two strategies specified within the second set of braces: (2.1) with probability 1/2, call if her card is 4 and fold if her cards are 1, 2, or 3; and (2.2) with probability 1/2, call if her cards are 2 or 4 and fold if her cards are 1 or 3.

Fast LPs for mixed NE

The NE can be considered from another point of view, whereby focusing on the card each player receives can reduce the number of constraints from exponential to linear. With this formulation, we can handle more than 200 cards now.

A strategy for Player I is given by a vector $\mathcal{P} = [p_1, \dots, p_n]$ that tells him: if his card is i , bet with probability p_i , and check with probability $1 - p_i$.

A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \dots, q_n]$ that tells her: if her card is j , call with probability q_j , and fold with probability $1 - q_j$.

Before we discuss the Fast LP formulation, let's mention that given card-by-card strategies, \mathcal{P} and \mathcal{Q} , it is easy to compute the *expected payoff* (for Player I). This is implemented in procedure `PayOffP1P2(n, b, P1, P2)`, as a bilinear form in the p_i 's and q_j 's:

$$\begin{aligned} & \frac{1}{n(n-1)} \left(\sum_{i=1}^n \sum_{j=1}^{i-1} (1-p_i) - \sum_{i=1}^n \sum_{j=i+1}^n (1-p_i) + \sum_{i=1}^n \sum_{j=1}^{i-1} p_i(1-q_j) \right. \\ & \left. + \sum_{i=1}^n \sum_{j=i+1}^n p_i(1-q_j) + (b+1) \sum_{i=1}^n \sum_{j=1}^{i-1} p_i q_j - (b+1) \sum_{i=1}^n \sum_{j=i+1}^n p_i q_j \right). \end{aligned}$$

Let's now get back to the Fast LP for Player I, which contains two sets of constraints. Each set corresponds to the expected payoff (over distribution \mathcal{P}), conditioned on the card that Player II has and whether she calls or folds:

$$\begin{aligned} & \text{Maximize } \frac{1}{n} \sum_{j=1}^n v_j \\ \text{s.t. } & \frac{1}{n-1} \sum_{i \neq j} (\text{Call}(i, j, b+1) \cdot p_i + \text{Call}(i, j, 1) \cdot (1-p_i)) \geq v_j \quad j = 1, \dots, n \text{ (Player II calls)} \\ & \frac{1}{n-1} \sum_{i \neq j} (p_i + \text{Call}(i, j, 1) \cdot (1-p_i)) \geq v_j \quad j = 1, \dots, n \text{ (Player II folds)} \\ & 0 \leq p_i \leq 1 \quad i = 1, \dots, n, \end{aligned} \tag{VN-I}$$

where the procedure `Call(i, j, R)` is defined based on whether the card i is larger than card j or not:

$$\text{Call}(i, j, R) = \begin{cases} R & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}$$

Similarly, for the Fast LP for Player II, the constraints are calculated based on the expected loss (over distribution \mathcal{Q}), conditioned on the card that Player I has and whether

he raises or checks:

$$\begin{aligned}
 & \text{Minimize } \frac{1}{n} \sum_{i=1}^n v_i \\
 \text{s.t. } & \frac{1}{n-1} \sum_{j \neq i} (\text{Call}(i, j, b+1) \cdot q_j + (1 - q_j)) \leq v_i \quad i = 1, \dots, n \text{ (Player I raises)} \\
 & \frac{1}{n-1} \sum_{j \neq i} \text{Call}(i, j, 1) \leq v_i \quad i = 1, \dots, n \text{ (Player I checks)} \\
 & 0 \leq q_j \leq 1 \quad j = 1, \dots, n. \tag{VN-II}
 \end{aligned}$$

The procedure for the Fast LPs is `vnMNE(n, b)`. Now things get interesting sooner. Already with three cards, we have bluffing!

With bet size 1, typing `lprint(vnMNE(3, 1))`; outputs:

```
[1/18, .55555555556e-1, [1/3, 0, 1], [0, 1/3, 1]] .
```

Translation:

- The value of the game is $\frac{1}{18}$.
- Its value in decimals is 0.055555....
- Player I's strategy is: If your card is 1, bet with probability $\frac{1}{3}$ and check with probability $\frac{2}{3}$. If your card is 2 then **definitely check**, while if your card is 3 then you should **definitely bet**.
- Player II's strategy is: If your card is 1, **definitely fold**, if your card is 2, call with probability $\frac{1}{3}$ and fold with probability $\frac{2}{3}$, while if your card is 3 then **definitely call**.

So already with three cards, Player I should sometimes bluff if his card is 1, but only with probability $\frac{1}{3}$.

The output file <https://sites.math.rutgers.edu/~zeilberg/tokhniot/oFinitePoker3.txt> contains one mixed NE for each of the cases n (size of the deck) from 2 to 40, and b (size of the bet) from 1 to 10.

The verbose form of `vnMNE(n, b)`; is `vnMNEv(n, b)`; spelling out the advice.

Note that a pure NE is also a mixed one, and indeed sometimes we get pure NEs. For example,

`lprint(vnMNE(9, 2))`; gives:

```
[1/9, .1111111111, [1, 0, 0, 0, 0, 0, 0, 1, 1], [0, 0, 0, 0, 0, 1, 1, 1, 1], 1, 1]].
```

Translation:

- The value of the game is $\frac{1}{9}$.
- The value of the game in floating-point is 0.11111111....
- Player I: Bet iff your card is in {1, 8, 9}.
- Player II: Call iff your card is in {6, 7, 8, 9}.

holds card j and the proposed $s \in Y$; otherwise, she folds.” The total number of these constraints amounts to $n \cdot 2^b$.

For example, when $b = 4$,

$$\mathcal{P}(S_4) = \{\{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}, \\ \{0, 3, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, \{0, 2, 3, 4\}, \{0, 1, 2, 3, 4\}\}.$$

With this setup, we derive the following LP:

$$\begin{aligned} & \text{Maximize } \frac{1}{n} \sum_{j=1}^n v_j \\ \text{s.t. } & \frac{1}{n-1} \sum_{i \neq j} \left(\sum_{s \in Y} \text{Call}(i, j, s+1) \cdot p_i[s] + \sum_{s \in (S_b \setminus Y)} p_i[s] \right) \geq v_j \quad \underbrace{j = 1, \dots, n; Y \in \mathcal{P}(S_b)}_{\text{total } n \cdot 2^b \text{ constraints}} \\ & \sum_{s=0}^b p_i[s] = 1, \quad i = 1, \dots, n \\ & p_i[s] \geq 0, \quad s = 0, \dots, b; i = 1, \dots, n \end{aligned} \tag{DJN-I}$$

Player II’s loss minimization

While Player’s II’s strategy is also an $n \times (b+1)$ matrix, formulating the LP is much simpler. Let’s denote the matrix by $(q_j[s])$ where $q_j[s]$ is the probability of calling if her card is j and the bet proposed by Player I is s (and as usual $1 - q_j[s]$ is the corresponding probability of folding). In this case, there are a total of $n(b+1)$ constraints (not exponential as in the case of Player I). Also, the LP formulation straightforwardly extends from (VN-II):

$$\begin{aligned} & \text{Minimize } \frac{1}{n} \sum_{i=1}^n v_i \\ \text{s.t. } & \frac{1}{n-1} \sum_{j \neq i} (\text{Call}(i, j, s+1) \cdot q_j[s] + (1 - q_j[s])) \leq v_i \quad \underbrace{s = 0, \dots, b; i = 1, \dots, n}_{\text{total } n(b+1) \text{ constraints}} \\ & q_j[0] = 1 \quad j = 1, \dots, n \\ & 0 \leq q_j[s] \leq 1 \quad s = 0, \dots, b; j = 1, \dots, n. \end{aligned} \tag{DJN-II}$$

These LPs are implemented in procedure `djnMNE(n, b)`; and the verbose version is `djnMNEv(n, b)`;

We noticed that for any given n , there exists a maximal bet size after which the game has the same value. The output file

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/oFinitePoker4a.txt> contains one mixed NE for $1 \leq n \leq 14$ and for all b until it ‘saturates’. As n grows larger, and b reaches its saturation value, the value of the game seems to converge to the DJ Newman ‘continuous’ value $\frac{1}{7}$.

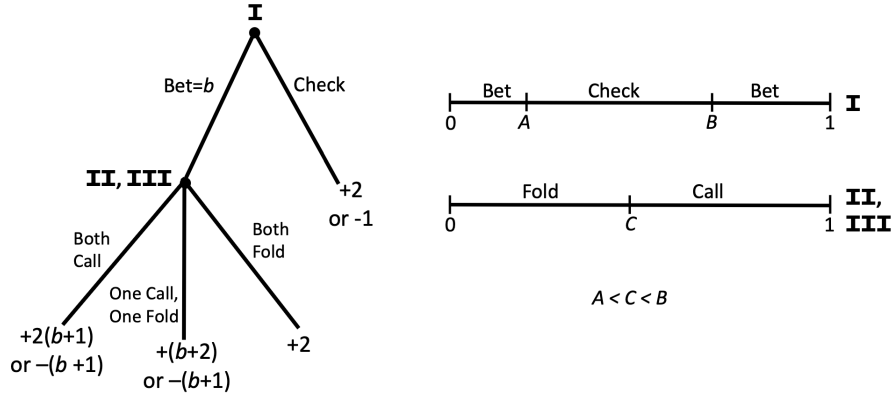


Figure 2: Three-player poker. Left: The betting tree. Right: Conjectured Nash equilibrium strategies for a continuous deck.

4 Three-player Poker Game

As early as 1950, future Economics Nobelists, John Nash and Lloyd Shapley [4], pioneered the analysis of a three-player poker game. They explored a simplified version where the deck contains only two kinds of cards, High and Low, in equal numbers. However, today, eighty years after von Neumann’s 1944 analysis of poker, the dynamics of the three-player game therein remain unexplored. We now take the opportunity to analyze these dynamics in both their finite and infinite versions.

Finite deck

The three players each put 1 dollar into the pot. Player I acts first, choosing either to check or to bet a fixed integer amount $b > 0$. If Player I checks, the three hands are immediately compared, and the player with the highest hand wins the pot. However, if Player I bets, Players II and III have two choices: call or fold. The reader is invited to refer to the left panel of Figure 2, which depicts the betting tree for three players. (The right panel shows the conjectured Nash equilibrium strategies to be used in the next section for the continuous version of the game.)

Assume we are given three-dimensional payoff matrices $(M_l, l = 1, 2, 3)$ for the three players:

$$M_l = (m_{ijk}^l),$$

where $i, j, k = 1, 2, \dots, 2^n$.

While its counterpart two-player game can be solved using linear programming, here we require nonlinear programming (NLP) [2]. The NLP formulation for the three-player game closely follows the LP model for the two players discussed in the previous section. Each player aims to minimize their expected loss, or the expected gain of the other players. For instance, given Player I’s payoff matrix M_1 , the other two players attempt to minimize the maximum potential loss incurred due to Player I’s choices. This involves constraints that utilize matrix M_1 and the probability distributions $\mathbf{y} = (y_1, \dots, y_{2^n})$ and $\mathbf{z} = (z_1, \dots, z_{2^n})$ of Players II and III. These are embedded in the first set of constraints in the NLP formulation, which we will now formulate.

The Slow NLP for three players is given by:

$$\begin{aligned}
& \text{Minimize } \sum_{l=1}^3 v^l \\
& \text{s.t. } \sum_{j,k=1}^{2^n} m_{ijk}^1 \cdot y_j \cdot z_k \leq v^1 \quad \text{for } i = 1, 2, \dots, 2^n \\
& \quad \sum_{i,k=1}^{2^n} m_{ijk}^2 \cdot x_i \cdot z_k \leq v^2 \quad \text{for } j = 1, 2, \dots, 2^n \\
& \quad \sum_{i,j=1}^{2^n} m_{ijk}^3 \cdot x_i \cdot y_j \leq v^3 \quad \text{for } k = 1, 2, \dots, 2^n \\
& \quad \sum_{i=1}^{2^n} x_i = 1, \quad \sum_{j=1}^{2^n} y_j = 1, \quad \sum_{k=1}^{2^n} z_k = 1 \\
& \quad x_i, y_j, z_k \geq 0 \quad \text{for } i, j, k = 1, 2, \dots, 2^n.
\end{aligned}$$

Note that if there are only two players, z_k in the above NLP formulation disappears, and the constraint functions become linear in the variables x_i and y_j . Thus, the problem can be decomposed into two separate LP (primal-dual) problems, as discussed earlier.

Programs related to three players can be found at <https://sites.math.rutgers.edu/~zeilberg/tokhniot/ThreePersonPoker.txt>. The Slow NLP is implemented in the procedure MNE(n, b), which is, of course, very slow even for small n .

We now shift our focus to the Fast NLP formulation for three players, which aligns with the Fast LP formulation for two players, considering on the card each player receives. Recall a strategy for Player I is given by a vector $\mathcal{P} = [p_1, \dots, p_n]$, indicating that if his card is i , he bets with probability p_i , and checks with probability $1 - p_i$. A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \dots, q_n]$, indicating that if her card is j , she calls with probability q_j , and folds with probability $1 - q_j$. Similarly, a strategy for Player III is represented by a vector $\mathcal{R} = [r_1, \dots, r_n]$, following the same interpretation as Player II.

We first define two procedures:

- Call2 is used to calculate the payoff if either Player II or Player III decides to fold, leaving only two players (one of whom is Player I) to compare their cards. Let us assume that Player III folds. Then,

$$\text{Call2}(i, j, R) = \begin{cases} R + 1 & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}$$

- Call3 is used to calculate the payoff when all the three players are comparing their cards:

$$\text{Call3}(i, j, k, R) = \begin{cases} 2R & \text{if } i > j \text{ and } i > k \\ -R & \text{if } i < j \text{ or } i < k. \end{cases}$$

The Fast NLP contains three sets of constraints, one set for each player, corresponding to the expected payoff over the pairs of distributions $\mathcal{Q} - \mathcal{R}$, $\mathcal{P} - \mathcal{R}$, or $\mathcal{P} - \mathcal{Q}$. For each

player $l = 1, 2, 3$, there are two sets of constraints depending on the card that Player l has and whether they follow their first strategy or the second strategy:

$$\text{Minimize } \frac{1}{n} \sum_{c=1}^n v_c^1 + \frac{1}{n} \sum_{c=1}^n v_c^2 + \frac{1}{n} \sum_{c=1}^n v_c^2$$

subject to

$$\frac{1}{(n-1)(n-2)} \sum_{j \neq i} \sum_{k \neq i, j} \text{Call13}(i, j, k, 1) \leq v_i^1 \quad i = 1, \dots, n \quad (\text{Player I checks})$$

$$\begin{aligned} & \frac{1}{(n-1)(n-2)} \left(\sum_{j \neq i} \sum_{k \neq i, j} \text{Call13}(i, j, k, b+1) \cdot q_k \cdot r_k \right. \\ & + \text{Call12}(i, j, b+1) \cdot q_j \cdot (1-r_k) + \text{Call12}(i, k, b+1) \cdot (1-q_j) \cdot r_k \\ & \left. + 2(1-q_j) \cdot (1-r_k) \right) \leq v_i^1 \quad i = 1, \dots, n \quad (\text{Player I bets}) \end{aligned}$$

$$\frac{1}{(n-1)(n-2)} \sum_{i \neq j} \sum_{k \neq i, j} (-p_i + \text{Call13}(j, i, k, 1) \cdot (1-p_i)) \leq v_j^2 \quad j = 1, \dots, n \quad (\text{Player II folds})$$

$$\begin{aligned} & \frac{1}{(n-1)(n-2)} \left(\sum_{i \neq j} \sum_{k \neq i, j} \text{Call13}(j, i, k, b+1) \cdot p_i \cdot r_k \right. \\ & \left. + \text{Call12}(j, i, b+1) \cdot p_i \cdot (1-r_k) + \text{Call13}(j, i, k, 1) \cdot (1-p_i) \right) \leq v_j^2 \quad j = 1, \dots, n \quad (\text{Player II calls}) \end{aligned}$$

$$\frac{1}{(n-1)(n-2)} \sum_{i \neq k} \sum_{j \neq i, k} (-p_i + \text{Call13}(k, i, j, 1) \cdot (1-p_i)) \leq v_k^3 \quad k = 1, \dots, n \quad (\text{Player III folds})$$

$$\begin{aligned} & \frac{1}{(n-1)(n-2)} \left(\sum_{i \neq k} \sum_{j \neq i, k} \text{Call13}(k, i, j, b+1) \cdot p_i \cdot q_j \right. \\ & \left. + \text{Call12}(k, i, b+1) \cdot p_i \cdot (1-q_j) + \text{Call13}(k, i, j, 1) \cdot (1-p_i) \right) \leq v_k^3 \quad k = 1, \dots, n \quad (\text{Player III calls}) \end{aligned}$$

$$0 \leq p_i, q_j, r_k \leq 1 \quad i, j, k = 1, \dots, n.$$

The Fast NLP for three players is implemented in procedure `FastMNE(n, b)`, which returns one solution to mixed NE. Here, we assume that Players II and III adopt identical strategies. For example, with bet size 1, typing

```
M := FastMNE(4,1):
[ListCF(M[1]),ListCF(M[2]),ListCF(M[3]),ListCF(M[4])];
gives
[[0, 1/24, -1/48, -1/48], [2/3, 0, 0, 1], [0, 0, 1/4, 1], [0, 0, 1/4, 1]].
```

Translation:

- The value of the game (for Player 1) is $\frac{1}{24}$, while for Players II and III are $-\frac{1}{48}$ each.
- Player I's strategy is: If your card is 1, bet with probability of $\frac{2}{3}$ and check with probability $\frac{1}{3}$. If your card is 2 or 3, then definitely checks; if your card is 4, definitely bet.
- Player II's and III's strategies are: If their card is 1 or 2, they definitely fold. If their card is 3, they call with probability of $\frac{1}{4}$ and fold with probability $\frac{3}{4}$. If their card is 4, they definitely call.

Another example,

```
M := FastMNE(10,2):
[ListCF(M[1]),ListCF(M[2]),ListCF(M[3]),ListCF(M[4])];
produces
[[0, 106/1125, -53/1125, -53/1125], [16/19, 0, 0, 0, 0, 0, 0, 0, 0, 1],
[0, 0, 0, 0, 0, 0, 3/25, 1, 1, 1], [0, 0, 0, 0, 0, 0, 3/25, 1, 1, 1]],
which we leave for the reader to interpret.
```

The verbose form of `FastMNE(n,b);` is `FastMNEVerbose(n,b);`, spelling out the advice. The output file <https://sites.math.rutgers.edu/~zeilberg/tokhniot/oThreePersonPoker1.txt> contains one mixed NE for each of the cases n (size of the deck) from 5 to 15, and b (size of the bet) from 1 to 3.

Extension of von Neumann's continuous model to three players

As you may recall, we introduced this work with von Neumann's concept of an uncountably infinite deck, contrasting it with the finite nature of real-world card games. We proceeded by solving the finite deck games for two players and extended our analysis to include three players. Now that we have solved the finite version, the solutions effortlessly transition us to the continuous version of the three-player game, as we will demonstrate—a fitting conclusion to our study.

As in the von Neumann model for two players, each of the three players contributes 1 dollar to the pot and receives independent uniform(0,1) hands. As a reminder, Player I has the option to check or bet a fixed amount b , while Players II and III can only call or fold. The betting tree remains the same as that of the finite deck model. The conjectured Nash equilibrium strategies for three players, guided by the data generated from the finite deck model, are illustrated in the right panel of Figure 2. Please scroll up a few pages.

We now determine the NE strategies.

Our advice

For numbers A, B, C , yet to be determined,

- Player I: If $0 < x < A$ or $B < x < 1$ he should **bet**, otherwise **check**.
- Players II and III: If $0 < y < C$ they should **fold**, otherwise **call**.

Assume $0 < A < C < B$. To determine the cut points A, B and C we solve three indifference equations as follows.

1. For Player I to be indifferent at A :

(a) If Player I checks at $x = A$, his expected payoff is

$$\int_0^A \int_0^A 2dzdy + \int_0^A \int_A^1 -1dzdy + \int_A^1 \int_0^1 -1dzdy$$

(b) If Player I bets at $x = A$, his expected payoff is

$$\int_0^C \int_0^C 2dzdy + \int_0^C \int_C^1 -(b+1)dzdy + \int_C^1 \int_0^1 -(b+1)dzdy$$

Equating the two expressions above yields the following equation:

$$3A^2 - 1 = 3C^2 + bC^2 - b - 1. \quad (\text{Eq. A})$$

2. For Player I to be indifferent at B :

(a) If Player I checks at $x = B$, his expected payoff is

$$\int_0^A \int_0^A 2dzdy + \int_0^A \int_A^1 -1dzdy + \int_A^1 \int_0^1 -1dzdy$$

(b) If Player I bets at $x = B$, his expected payoff is

$$\begin{aligned} & \int_0^C \int_0^C 2dzdy + \int_0^C \int_C^B (b+2)dzdy + \int_C^B \int_0^C (b+2)dzdy + \int_C^B \int_C^B 2(b+1)dzdy \\ & + \int_B^1 \int_0^B -(b+1)dzdy + \int_0^B \int_B^1 -(b+1)dzdy + \int_B^1 \int_B^1 -(b+1)dzdy \end{aligned}$$

Equating the two expressions above yields the following equation:

$$3B^2 - 1 = -2bCB + 3bB^2 + 3B^2 - b - 1. \quad (\text{Eq. B})$$

3. For Player II (or Player III) to be indifferent at C :

(a) If Player II folds at $y = C$, her expected payoff is

$$\int_0^A \int_0^1 -1dzdx + \int_B^1 \int_0^1 -1dzdx$$

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The authors are pictured in front of John von Neumann’s former residence at 26 Westcott Road in Princeton. The photo, taken on May 31, 2024, was captured by Karen Reid, the current owner, who kindly permitted us to take it.