

# FORMULÆ FOR THE NUMBER OF PARTITIONS OF $n$ INTO AT MOST $m$ PARTS (USING THE QUASI-POLYNOMIAL ANSATZ)

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## Abstract

The purpose of this short article is to announce, and briefly describe, a Maple package, PARTITIONS, that (inter alia) *completely automatically* discovers, and then proves, explicit expressions (as sums of quasi-polynomials) for  $p_m(n)$  for any desired  $m$ . We do this to demonstrate the power of “rigorous guessing” as facilitated by the quasi-polynomial ansatz.

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## 1. Introduction

Recall that a *partition* of a non-negative integer  $n$  is a non-increasing sequence of positive integers  $\lambda_1 \dots \lambda_m$  that sum to  $n$ . For example the integer 5 has the following seven partitions:  $\{5, 41, 32, 311, 221, 2111, 11111\}$ . The **bible** on partitions is George Andrews' *magnum opus* [1].

We denote by  $p_m(n)$  the number of partitions of  $n$  into at most  $m$  parts. By a classic theorem [1, p. 8, Thm. 1.4],  $p_m(n)$  also equals the number of partitions of  $n$  into parts that are at most  $m$ . There is an extensive literature concerning formulæ for  $p_m(n)$ , including contributions by Cayley, Sylvester, Glaisher, and Gupta. For additional references and historical notes, see George Andrews' fascinating article [2, §3] and Gupta's *Tables* [7, pp. i–xxxix]. For an exhaustive history through 1920, see Dickson [4, Ch. 3].

More recently, George Andrews' student, Augustine O. Munagi, developed a beautiful theory of so-called  $q$ -partial fractions [9], where the denominators in the decomposition are always expressions of the form  $(1 - q^r)^s$ , rather than powers of cyclotomic polynomials as is the case with the ordinary partial fraction decomposition. Accordingly, formulæ for  $p_m(n)$  derived from the  $q$ -partial fraction decomposition of the generating function are most naturally expressed in terms of binomial coefficients.

It is well-known and easy to see that for any  $m$ ,  $p_m(n)$  is a sum of *quasi-polynomials of periods*  $1, 2, 3, \dots, m$ . A *quasi-polynomial of period*  $r$  is a function  $f(n)$  on the integers such that there exist  $r$  polynomials  $P_1(n), P_2(n), \dots, P_r(n)$  such that  $f(n) = P_i(n)$  if  $n \equiv i \pmod{r}$ . We represent such a quasi-polynomial as a list  $[P_1(n), \dots, P_r(n)]$ .

## 2. The PARTITIONS Maple package

### 2.1. Overview

The purpose of this short article is to announce, and briefly describe, a Maple package, PARTITIONS, that *completely automatically* discovers, and then proves, explicit expressions (as sums of quasi-polynomials) for  $p_m(n)$  for any desired  $m$ . So far we only bothered to derive the formulæ for  $1 \leq m \leq 70$ , but one can easily go far beyond.

**Not only that**, we can, more generally, derive (and prove!), completely automatically, expressions, as sums of quasi-polynomials, for the number of ways of making change for  $n$  cents in a country whose coins have denominations of any given set of positive integers.

**Not only that**, we can, derive (and prove!), completely automatically, expressions (as sums of quasi-polynomials) for  $D_k(n)$ , the number of partitions of  $n$  whose *Durfee square* has size  $k$ , for any desired, (numeric) positive integer  $k$ . (Recall that the size of the Durfee square of a partition  $\lambda_1 \dots \lambda_m$  is the largest  $k$  such that  $\lambda_k \geq k$ .)

**Not only that**, we (or rather our computers (and of course yours, if it has Maple and is loaded with our package)) can derive **asymptotic expressions**, to *any desired order*, for both  $p_m(n)$  and  $D_k(n)$ . As far as we know the formula for  $D_k(n)$  is brand-new, and the previous attempts for the asymptotic formula for  $p_m(n)$  by humans G.J. Rieger [12] and E.M. Wright [14] (of Hardy-and-Wright fame) only went as far as  $O(n^{-2})$  and  $O(n^{-4})$  respectively. We go all the way to  $O(n^{-100})!$  (and of course can easily go far beyond).

**Not only that**, we implement George Andrews' ingenious way ([2, sec. 3]) to convert any quasi-polynomial to a polynomial expression where one is also allowed to use the integer-part function  $\lfloor n \rfloor$ . This enabled our computers to find Andrews-style expressions for  $p_m(n)$  for  $1 \leq m \leq 70$ .

All these feats (and more!) are achieved by the **Maple package** PARTITIONS.

## 2.2. Using the PARTITIONS package

In order to use PARTITIONS, you must have Maple<sup>TM</sup> installed on your computer. Then download the file:

<http://www.math.rutgers.edu/~zeilberg/tokhniot/PARTITIONS> and save it as PARTITIONS. Then launch Maple, and at the prompt, enter:

read PARTITIONS:

and follow the on-line instructions. Let's just highlight the most important procedures.

AS100( $m, n$ ): shows the *pre-computed* first 100 terms of the asymptotic expression, in  $n$ , of  $p_m(n)$  for *symbolic*  $m$ .

ASD80( $k, n$ ): shows the *pre-computed* first 80 terms of the asymptotic expression, in  $n$ , of  $D_k(n)$  for *symbolic*  $k$ .

BuildDBpmn( $n, M$ ): inputs a symbol  $n$  and a positive integer  $M$  and outputs a list of size  $M$  whose  $i$ -th entry is an expression for  $p_i(n)$  as a sum of  $i$  quasi-polynomials

DiscoverAS( $m, n, k$ ): discovers the asymptotic expansion to order  $k$  of  $p_m(n)$  (the number of partitions of  $n$  into at most  $m$  parts) for large  $n$  and fixed, but *symbolic*,  $m$ .

DiscoverDAS( $k, n, r$ ): discovers the asymptotic expansion to order  $r$  of  $D_k(n)$  (the number of partitions of  $n$  whose Durfee square has size  $k$ ) for large  $n$  and fixed, but *symbolic*  $k$ .

Durfee( $k, n$ ): discovers (rigorously!) the quasi-polynomial expression, in  $n$ , for  $D_k(n)$ , for any desired positive integer  $k$ . It is extremely fast for small  $k$ , but of course gets slower as  $k$  gets larger.

DurfeePC( $k, n$ ): does the same thing (much faster, of course!) using the pre-computed expressions of Durfee( $k, n$ ); for  $k \leq 40$ .

EvalQPS( $L, n, n_0$ ): evaluates the sum of the quasi-polynomials in the variable  $n$  given in the list  $L$  at  $n = n_0$ .

HRR( $n, T$ ): evaluates in floating point the sum of the first  $T$  terms of the Hardy-Ramanujan-Rademacher formula for  $p(n)$ , the number of unrestricted partitions of  $n$ :

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k \geq 1} \sqrt{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e^{\pi i(s(h,k) - 2nh/k)} \frac{d}{dn} \left( \frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}} \left(n - \frac{1}{24}\right)\right)}{\sqrt{n - \frac{1}{24}}} \right),$$

where  $s(h, k) = \sum_{j=1}^{k-1} \left( \frac{j}{k} - \lfloor \frac{j}{k} \rfloor - \frac{1}{2} \right) \left( \frac{hj}{k} - \lfloor \frac{hj}{k} \rfloor - \frac{1}{2} \right)$  is the Dedekind sum.

Please be warned that for larger  $n$  you need to increase `Digits`. In order to get reliable results you may want to use procedure `HRRr(n,T,k)`.

pmn( $m, n$ ): discovers (rigorously!) the quasi-polynomial expression, in  $n$ , for  $p_m(n)$ , for any desired positive integer  $m$ . It is extremely fast for small  $m$ , but of course gets slower as  $m$  gets larger.

`pmnPC(m, n)`: does the same thing (much faster, of course!) using the pre-computed expressions of `pmn(m, n)`; for  $m \leq 70$ .

`pmnAndrews(m, n)`: discovers (rigorously!) the Andrews-style expression, in  $n$ , for  $p_m(n)$  for any desired positive integer  $m$ . Instead of using quasi-polynomials explicitly (that some humans find awkward), it uses the integer-part function  $\lfloor n \rfloor$ , denoted by `trunc(n)` in Maple.

`pn(n)`: the number of partitions of  $n$ ,  $p(n)$ , using Euler's recurrence. It is useful for checking, since  $p_n(n) = p(n)$ .

`pnSeq(N)`: the list of the first  $N$  values of  $p(n)$ . The output of `pnSeq(50000)` can be gotten from

<http://www.math.rutgers.edu/~zeilberg/tokhniot/oPARTITIONS9> where this list of 50000 terms is called `pnTable`.

`pSn(S, n, K)`: the more general problem where the parts are drawn from the list  $S$  of positive integers. It outputs an explicit expression, as a sum of quasi-polynomials, for  $p_S(n)$ , the number of integer partitions of  $n$  whose parts are drawn from the finite list of positive integers  $S$ .  $K$  is a guessing parameter, that should be made higher if the procedure returns FAIL.

`pmnNum(m, n0)`: like `pmn(m, n)`; but for both *numeric*  $m$  and  $n0$ . The output is a number. For  $m \leq 70$  it is extremely fast, since it uses the *pre-computed* values of  $p_m(n)$  gotten from `pmnPC(m, n)`; . For example to get the number of integer partitions of a googol ( $10^{100}$ ) into at most 60 parts, you would get, in 0.02 seconds, the 5783-digit integer, by simply typing

`pmnNum(60, 10**100);` .

One of us (DZ) posed this as a 100-dollar challenge to the users of the very useful Mathoverflow forum. This was taken-up, successfully, by user *joro* [5], whose computer did it correctly in about 2 hours, using PARI. User *joro* generously suggested that instead of sending him a check, DZ should donate it in *joro*'s honor, to a charity of DZ's choice, and the latter decided on the Wikipedia Foundation.

**Sample input and output** can be gotten from the "front" of this article:

<http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/pmn.html> .

### 3. Methodology: Rigorous Guessing

The idea of deriving formulæ for  $p_m(n)$  and  $p_S(n)$  with the aid of the partial fraction decomposition of the generating function dates back at least to Cayley [3]. We ask Maple to convert the generating function

$$\sum_{n \geq 0} p_m(n) q^n = \frac{1}{(1-q)(1-q^2) \cdots (1-q^m)}$$

or in the case of  $p_S(n)$ , where  $S = \{s_1, s_2, \dots, s_j\}$ ,

$$\sum_{n=0}^{\infty} p_S(n) q^n = \frac{1}{(1-q^{s_1})(1-q^{s_2}) \cdots (1-q^{s_j})}$$

into partial fractions. Then for each piece, Maple finds the first few terms of the Maclaurin expansion, and then fits the data with an appropriate quasi-polynomial using *undetermined*

coefficients. The output is the sum of these quasi-polynomials. See the source-code for more details.

**Example.** Consider the case  $m = 4$ . We have Maple calculate that

$$\begin{aligned} \sum_{n \geq 0} p_4(n)q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &= \frac{17/72}{1-q} + \frac{59/288}{(1-q)^2} + \frac{1/8}{(1-q)^3} + \frac{1/24}{(1-q)^4} + \frac{1/8}{1+q} + \frac{1/32}{(1+q)^2} + \frac{(1+q)/9}{1+q+q^2}. \end{aligned} \quad (1)$$

At this point we could, as Cayley did, expand each term as a series in  $q$ , collect like terms, and then the coefficient of  $q^n$  will be a formula for  $p_4(n)$ , but why bother? From Sylvester [13] and Glaisher [6], we know that

$$p_4(n) = \sum_{j=1}^4 W_j(n),$$

where each  $W_j(n)$  is a quasi-polynomial  $[P_{j1}(n), P_{j2}(n), \dots, P_{jj}(n)]$  of period  $j$ . Further,  $W_j(n)$  is of degree  $\lfloor \frac{n-j}{j} \rfloor$ , and arises from those terms of (1) with denominator a power of the  $j$ -th cyclotomic polynomial. Instead, let us allow Maple to guess the correct quasi-polynomials: We know *a priori* that  $W_1(n)$  is of the form  $[a_0 + a_1n + a_2n^2 + a_3n^3]$  and let Maple calculate the (beginning of the) Maclaurin series for the terms of (1) that contribute to  $W_1(n)$ :

$$\frac{17/72}{1-q} + \frac{59/288}{(1-q)^2} + \frac{1/8}{(1-q)^3} + \frac{1/24}{(1-q)^4} = \frac{175}{288} + \frac{19}{16}q + \frac{581}{288}q^2 + \frac{113}{36}q^3 + O(q^4).$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1^0 & 1^1 & 1^2 & 1^3 \\ 2^0 & 2^1 & 2^2 & 2^3 \\ 3^0 & 3^1 & 3^2 & 3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 175/288 \\ 19/16 \\ 581/288 \\ 113/36 \end{bmatrix},$$

which immediately implies that

$$W_1(n) = \left[ \frac{1}{144}n^3 + \frac{5}{48}n^2 + \frac{15}{32}n + \frac{175}{288} \right].$$

Similarly, for  $W_2(n)$ , which must be of the form

$$[a_1 + a_3n, a_0 + a_2n],$$

we find

$$\frac{1/8}{1+q} + \frac{1/32}{(1+q)^2} = \frac{5}{32} - \frac{3}{16}q + \frac{7}{32}q^2 - \frac{1}{4}q^3 + O(q^4),$$

so that

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5/32 \\ 7/32 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} -3/16 \\ -1/4 \end{bmatrix},$$

and thus

$$W_2(n) = \left[ -\frac{5}{32} - \frac{n}{32}, \frac{5}{32} + \frac{n}{32} \right].$$

Analogous reasoning yields  $W_3(n) = [0, -\frac{1}{9}, \frac{1}{9}]$  and  $W_4(n) = [0, -\frac{1}{8}, 0, \frac{1}{8}]$ .

## 4. Conclusion

The present approach uses *very naïve* guessing to discover, and *prove* (rigorously!), formulas (or as Cayley and Sylvester would say, *formulae*) for the number of partitions of the integer  $n$  into at most parts  $m$  parts for  $m \leq 70$ , and of course, one can easily go far beyond. The core of the idea goes back to Arthur Cayley, and is familiar to any second-semester calculus student: partial fractions! But dear Arthur could only go so far, so his good buddy, James Joseph Sylvester, invented a sophisticated theory of “waves” [13] that facilitated hand calculations, which were later dutifully carried out by J. W. L. Glaisher in [6]. But, with modern computer algebra systems (Maple in our case), one can go much further by just using Cayley’s original ideas.

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