A (Human!) Proof of A Conjectured Triple Sum Identity Made By Juan Sebastián Pereyra

Doron ZEILBERGER[1]

In [P] the following conjecture was made

$$\sum_{i=1}^{n} \left[i^2 \sum_{k=0}^{m} \left(\binom{i-1}{m-k} \binom{n-i}{k} \right)^2 \right] +$$

$$2 \sum_{j=1}^{m} \left[\sum_{i=1}^{n-j} i(i+j) \left(\sum_{k=j}^{m} \binom{i+j-1}{k} \binom{n-(i+j)}{m-k} \binom{n-i}{m-k+j} \binom{i-1}{k-j} \right) \right]$$

is equal to

$$\binom{n+1}{m}^2 \left(\sum_{i=1}^{n-m} i^2 \right) \quad .$$

Of course the right side is $\binom{n+1}{m}^2 (n-m)(n-m+1)(2n-2m+1)/6$ (using $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$), and the left side can be written more symmetrically as

$$\sum_{i,j,k} i(i+j) \binom{i+j-1}{k} \binom{n-(i+j)}{m-k} \binom{n-i}{m-k+j} \binom{i-1}{k-j} \quad ,$$

where the summation range is over all triples (i,j,k) for which the binomial coefficients "make sense" i.e. whenever you have $\binom{r}{s}$ we have $0 \le s \le r$).

Writing $a = i+j$, (and leaving i as a discrete variable, but letting $j = a-i$), it is more convenient to evaluate

$$\sum_{a,k,i} ia \binom{a-1}{k} \binom{n-a}{m-k} \binom{n-i}{m-k+a-i} \binom{i-1}{k-a+i} \quad .$$

It is easily seen that this equals the *iterated* summation

$$\sum_{a=1}^{n} \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \sum_{i=a-k}^{a-k+m} ia \binom{a-1}{k} \binom{n-a}{m-k} \binom{n-i}{m-k+a-i} \binom{i-1}{k-a+i}$$

$$= \sum_{a=1}^{n} a \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-1}{k} \binom{n-a}{m-k} \sum_{i=a-k}^{a-k+m} i \binom{n-i}{m-k+a-i} \binom{i-1}{k-a+i} \quad .$$

[1] Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. `zeilberg at math dot rutgers dot edu` ,
`http://www.math.rutgers.edu/~zeilberg/` . March 5, 2013. Exclusively publised in the Personal Journal of Shalosh B. Ekhad and Doron Zeilberger.
`http://www.math.rutgers.edu/~zeilberg/pj.html` . Supported in part by the NSF.

Let's do the innermost sum first. Since $i\binom{i-1}{k-a+i} = (a-k)\binom{i}{a-k}$ the inner sum is

$$(a-k)\sum_{i=a-k}^{a-k+m}\binom{n-i}{m-k+a-i}\binom{i}{a-k} = (a-k)\binom{n+1}{m} \quad ,$$

by the celebrated Vandermonde-Chu (alias terminating Gauss's ${}_2F_1(1)$ identity).

Now let's do the middle sum

$$\binom{n+1}{m}\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)}(a-k)\binom{a-1}{k}\binom{n-a}{m-k} \quad .$$

Now

$$\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)}(a-k)\binom{a-1}{k}\binom{n-a}{m-k}$$

$$= a\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)}\binom{a-1}{k}\binom{n-a}{m-k} - \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)}k\binom{a-1}{k}\binom{n-a}{m-k} \quad .$$

The first sum is $a\binom{n-1}{m}$, again by Vandermonde, while for the second we replace

$$k\binom{a-1}{k}$$

by

$$(a-1)\binom{a-2}{k-1}$$

getting that it equals

$$(a-1)\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)}\binom{a-2}{k-1}\binom{n-a}{m-k} \quad ,$$

that again by Vandermonde, equals $(a-1)\binom{n-2}{m-1}$. So the inner and the middle $\sum$'s give the following expression in m, n, a:

$$\binom{n+1}{m}\left(a\binom{n-1}{m} - (a-1)\binom{n-2}{m-1}\right) \quad .$$

Finally to do the outermost sum, w.r.t. to a (don't forget the a in front that we took out since it didn't depend on i and k)

$$\binom{n+1}{m}\left(\binom{n-1}{m}\left(\sum_{a=1}^{n}a^2\right) - \binom{n-2}{m-1}\left(\sum_{a=1}^{n}a(a-1)\right)\right)$$

$$= \binom{n+1}{m}\left(\frac{(n-1)!}{m!(n-m-1)!}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{(n-2)!}{(m-1)!(n-m-1)!}\cdot\frac{(n-1)n(n+1)}{3}\right)$$

$$= \binom{n+1}{m}\cdot\left(\frac{(n+1)!}{m!(n-m-1)!}\cdot\frac{(2n+1)}{6} - \frac{(n+1)!}{m!(n-m-1)!}\cdot\frac{m}{3}\right)$$

$$= \binom{n+1}{m} \frac{(n+1)!}{m!(n-m-1)!} \cdot \left(\frac{2n+1}{6} - \frac{m}{3} \right)$$

$$= \binom{n+1}{m} \frac{(n+1)!}{m!(n-m-1)!} \cdot \frac{2n-2m+1}{6}$$

$$= \binom{n+1}{m} \cdot \frac{(n+1)!}{m!(n-m+1)!} \cdot \frac{(n-m)(n-m+1)(2n-2m+1)}{6}$$

$$= \binom{n+1}{m}^2 \frac{(n-m)(n-m+1)(2n-2m+1)}{6} \quad \square.$$

References

[P] Juan Sebastián Pereyra, *private communication*,
`http://www.math.rutgers.edu/~zeilberg/akherim/pereyraConj.pdf`