

**The Reciprocal of $\sum_{n \geq 0} a^n b^n$ for non-commuting a and b ,
Catalan numbers and non-commutative quadratic equations**

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Note: This article is accompanied by the Maple package NCFPS downloadable from
<http://www.math.rutgers.edu/~zeilberg/tokhniot/NCFPS>

The aim of this paper is to describe the inversion of the sum $\sum_{n \geq 0} a^n b^n$ where a and b are non-commuting variables as a formal series in a and b . We show that the inversion satisfies a non-commutative quadratic equation and that the number of certain monomials in its homogeneous components equals a Catalan number. We also study general solutions of similar quadratic equations.

1. Inverting $\sum_{n \geq 0} a^n b^n$.

Our goal is to find an inverse of the series $\sum_{n \geq 0} a^n b^n$ where a and b are non-commuting variables. The answer to this question is given by the following theorem.

Let a, b, x be (completely!) **non-commuting** variables (“indeterminates”). Define a sequence of polynomials $d_n(a, b, x)$ ($n \geq 1$) recursively as follows:

$$d_1(a, b, x) = 1 \quad , \quad (1a)$$

$$d_n(a, b, x) = d_{n-1}(a, b, x)x + \sum_{k=2}^{n-1} d_{n-k}(a, b, x) a d_k(a, b, x) b \quad (n \geq 2) \quad .(1b)$$

Also define the sequence of polynomials $c_n(a, b, x)$ as follows:

$$c_n(a, b, x) = a d_n(a, b, x) b \quad (n \geq 1) \quad .$$

Theorem 1:

$$1 - \sum_{n=1}^{\infty} c_n(a, b, ab - ba) = \left(\sum_{n \geq 0} a^n b^n \right)^{-1} .$$

It follows immediately that the number of monomials in a, b and x in the polynomial $d_n(a, b, x)$ is the $(n - 1)$ -th Catalan number. In particular, $d_1 = 1$, $d_2 = x$, $d_3 = x^2 + axb$,

$$d_4 = x^3 + ax^2b + axbx + xaxb + a^2xb^2,$$

$$\begin{aligned} d_5 = & x^4 + ax^2bx + axbx^2 + xaxbx + a^2xb^2x + x^2axb + axbaxb \\ & + xax^2b + xaxb^2 + ax^3b + a^2x^2b^2 + a^2xbxb + axaxb^2 + a^3xb^3. \end{aligned}$$

We will give an algebraic and a combinatorial proof of the theorem. A simple algebraic proof is based on two lemmas.

Lemma 2: Let S be a formal series in a and b such that $S = 1 + aSb$. Observe that the inverse of S is of the form $1 - C$ where $C = aDb$ and the series D satisfies the equation

$$D = 1 + D(x - ab) + DaDb \quad (2)$$

and $x = ab - ba$.

Proof: We are looking for the inverse of S in the form $1 - C$ where $C = aDb$.

We have

$$CS = (1 - S^{-1})S = S - 1 = aSb.$$

Hence

$$C(1 + aSb) = aSb,$$

$$C + CaSb = aSb,$$

$$aDb + aDbaSb = aSb.$$

So,

$$D + DbaS = S$$

and

$$D(1 + baS) = S$$

or

$$D(S^{-1} + ba) = 1.$$

It implies that

$$D(1 - C + ba) = 1$$

and

$$D = 1 + DaDb - Dba$$

which immediately implies equation (2).

Lemma 3: Let the degree of indeterminates a and b in equation (2) equal one and the degree of x equal two. Then the solution of equation (2) is given by formula

$$D = \sum_{n \geq 1} d_n(a, b, x)$$

where polynomials $d_n(a, b, x)$ satisfy equations (1).

Proof: Note that $D = \sum_{n=1}^{\infty} d_n$ where $d_n = d_n(a, b, x)$ are homogeneous polynomials in a and b of degree $2n - 2$, $n = 1, 2, \dots$

The terms of degree 0 and 2 are: $d_1 = 1$ and $d_2 = x$.

Take the term of degree $2n - 2$, $n \geq 3$:

$$\begin{aligned} d_n &= d_{n-1}(x - ab) + \sum_{k=1}^{n-1} d_{n-k} a d_k b = d_{n-1}(x - ab) + d_{n-1}ab + d_1 a d_{n-1}b + \sum_{k=2}^{n-2} d_{n-k} c_k = \\ &= d_{n-1}x + a d_{n-1}b + \sum_{k=2}^{n-2} d_{n-k} c_k. \end{aligned}$$

QED

Let $S = \sum_{n \geq 0} a^n b^n$. Then S satisfies equation $S = 1 + aSb$ and Theorem 1 follows from Lemmas 2 and 3.

Combinatorial Proof: Consider the set of *lattice walks* in the 2D rectangular lattice, starting at the origin, $(0, 0)$ and ending at $(n - 1, n - 1)$, where one can either make a *horizontal* step $(i, j) \rightarrow (i + 1, j)$, (weight a), a *vertical* step $(i, j) \rightarrow (i, j + 1)$, (weight b) or a diagonal step $(i, j) \rightarrow (i + 1, j + 1)$, (weight x), always staying in the region $i \geq j$, and where you can never have a horizontal step followed immediately by a vertical step. In other words, you may never venture to the region $i < j$, and you can never have the Hebrew letter Nun (alias the mirror-image of the Latin letter L) when you draw the path on the plane. The weight of a path is the product (in order!) of the weights of the individual steps.

For example, when $n = 2$ the only possible path is $(0, 0) \rightarrow (1, 1)$, whose weight is x .

When $n = 3$ we have two paths. The path $(0, 0) \rightarrow (1, 1) \rightarrow (2, 2)$ whose weight is x^2 and the path $(0, 0) \rightarrow (1, 0) \rightarrow (2, 1) \rightarrow (2, 2)$ whose weight is axb .

When $n = 4$ we have five paths:

The path $(0, 0) \rightarrow (1, 1) \rightarrow (2, 2) \rightarrow (3, 3)$ whose weight is x^3 ,

the path $(0, 0) \rightarrow (1, 0) \rightarrow (2, 1) \rightarrow (3, 2) \rightarrow (3, 3)$ whose weight is ax^2b ,

the path $(0, 0) \rightarrow (1, 0) \rightarrow (2, 1) \rightarrow (2, 2) \rightarrow (3, 3)$ whose weight is $axbx$,

the path $(0, 0) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow (3, 2) \rightarrow (3, 3)$ whose weight is $xaxb$, and

the path $(0, 0) \rightarrow (1, 0) \rightarrow (2, 0) \rightarrow (3, 1) \rightarrow (3, 2) \rightarrow (3, 3)$ whose weight is a^2xb^2 .

It is very well-known, and rather easy to see, that the number of such paths are given by the Catalan numbers $C(n-1)$, [2] <http://oeis.org/A000108>.

We claim that the *weight-enumerator* of the set of such walks equals $d_n(a, b, x)$. Indeed, since the walk ends on the diagonal, at the point $(n-1, n-1)$, the last step must be either a diagonal step

$$(n-2, n-2) \rightarrow (n-1, n-1) \quad ,$$

whose weight-enumerator, by the inductive hypothesis is $d_{n-1}(a, b, x)x$, or else let k be the smallest integer such that the walk passed through $(n-k-1, n-k-1)$ (i.e. the penultimate encounter with the diagonal). Note that k can be anything between 2 and $n-1$. The weight-enumerator of the set of paths from $(0, 0)$ to $(n-k-1, n-k-1)$ is $d_{n-k}(a, b, x)$, and the weight-enumerator of the set of paths from $(n-k-1, n-k-1)$ to $(n-1, n-1)$ that never touch the diagonal, is $ad_k(a, b, x)b$. So the weight-enumerator is $d_{n-k}(a, b, x)ad_k(a, b, x)b$ giving the above recurrence for $d_n(a, b, x)$.

It follows that $c_n(a, b, x) = ad_n(a, b, x)b$ is the weight-enumerator of all paths from $(0, 0)$ to (n, n) as above with the additional property that except at the beginning $((0, 0))$ and the end $((n, n))$ they always stay **strictly** below the diagonal.

Now what does $c_n(a, b, ab - ba)$ weight-enumerate? Now there is a new rule in Manhattan, “no shortcuts”, one may not walk diagonally. So every diagonal step $(i, j) \rightarrow (i+1, j+1)$ must decide whether

to go first horizontally, and then vertically $(i, j) \rightarrow (i+1, j) \rightarrow (i+1, j+1)$, replacing x by ab , or

to go first vertically, and then horizontally $(i, j) \rightarrow (i, j+1) \rightarrow (i+1, j+1)$, replacing x by $-ba$.

This has to be decided, independently for each of the diagonal steps that formerly had weight x . So a path with r diagonal steps gives rise to 2^r new paths with sign $(-1)^s$ where s is the number of places where it was decided to go through the second option.

So $c_n(a, b, ab - ba)$ is the weight-enumerators of pairs of paths $[P, K]$ where P is the original path featuring a certain number of diagonal steps r , and K is one of its 2^r “children”, paths with only horizontal and vertical steps, and weight $\pm \text{weight}(P)$, where we have a plus-sign if an even number of the r diagonal steps became *vertical-then-horizontal* (i.e. ba) and a minus-sign otherwise.

As we look at the weights of the children K , sometimes we have the same path coming from different parents. Let's call a pair $[P, K]$ *bad* if the path P has a “ ba ” *strictly-under* the diagonal, i.e. a “vertical step followed by a horizontal step” that does not touch the diagonal. Write K as $K = w_1(ba)^s w_2$ where w_1 does not have any sub-diagonal ba 's and s is as large as possible. Then the parent must be either of the form $P = W_1 x^s W_2$ where the x^s corresponds to the $(ba)^s$, or of the form $P' = W_1 b x^{s-1} a W_2$. In the former case attach $[W_1 x^s W_2, K]$ to $[W_1 b x^{s-1} a W_2, K]$ and in the latter case vice-versa. This is a weight-preserving and **sign-reversing** involution among the

bad pairs, so they all kill each other.

It remains to weight-enumerate the *good pairs*. It is easy to see that the good pairs are pairs $[P, K]$ where K has the form $K = a^{i_1}b^{i_1}a^{i_2}b^{i_2} \dots a^{i_s}b^{i_s}$ for some $s \geq 1$ and integers $i_1, \dots, i_s \geq 1$ summing up to n (this is called a *composition* of n). It is easy to see that for each such K , (coming from a good pair $[P, K]$) there can only be **one** possible *parent* P . The sign of a good pair

$$[P, a^{i_1}b^{i_1}a^{i_2}b^{i_2} \dots a^{i_s}b^{i_s}] \quad ,$$

is $(-1)^{s-1}$, since it touches the diagonal $s - 1$ times, and each of these touching points came from an x that was turned into $-ba$.

So $1 - \sum_{n=1}^{\infty} c_n(a, b, ab - ba)$ turned out to be the sum of all the weights of compositions (vectors of positive integers) (i_1, \dots, i_s) with the weight $(-1)^s a^{i_1}b^{i_1} \dots a^{i_s}b^{i_s}$ over *all* compositions, but the same is true of

$$\left(\sum_{n \geq 0} a^n b^n \right)^{-1} = \left(1 + \sum_{n \geq 1} a^n b^n \right)^{-1} = 1 + \sum_{s=1}^{\infty} (-1)^s \left(\sum_{n \geq 1} a^n b^n \right)^s .$$

QED!

2. Inversion of $1 - aDb$ in the general case.

The following is a variant of of path's model used in Section 1. Call *Dyck path* a path that starts at the origin, ends on the x -axis, that uses the steps $(1, 1)$ (denoted by a) and $(1, -1)$ (denoted by b), and that never goes below the x -axis. It is coded by a *Dyck word*, e.g. $aaababbabb$. Formally, a Dyck word has as many a 's than b 's, and each prefix of it has at least as many a 's as b 's.

If we replace, in each Dyck word, each occurrence of ab by a letter x , and sum all these words, then we obtain the series $D = \sum_{n \geq 1} d_n$ described in Section 1.

If we replace each ab by a letter x , except those at level 0, then we obtain the series

$$1 + aUb = 1 + \sum_{n \geq 1} au_n b.$$

For a series Z set $Z^* := (1 - Z)^{-1}$. Then

$$(aDb)^* = 1 + aUb.$$

Theorem 4. One has the equation

$$U = (1 + aUb)(1 + (x - ab + ba)U)$$

that completely defines U .

Proof: We have $(1 - aDb)^{-1} = 1 + aUb$, thus $1 - aDb = (-aUb)^*$.

The defining equation for D is

$$D = 1 + (x - ab + aDb)D \quad (3)$$

which is a symmetric version of equation (2); it follows from the Dyck path model, by writing $D = 1 + (x + a(D - 1)b)D$. Let $D = 1 + d_1(a, b, x) + d_2(a, b, x) + \dots$ and polynomials $d_n(a, b, x)$ that satisfy equations (1) without any assumptions on x .

We have

$$1 - aDb = (-aUb)^* = 1 - aUb + (aUb)^2 - (aUb)^3 + \dots$$

Therefore,

$$\begin{aligned} aDb &= aUb - (aUb)^2 + (aUb)^3 - \dots = a(U - UbaU + UbaUbaU - \dots)b \\ &= aU(1 - baU + (baU)^2 - \dots)b \end{aligned}$$

and

$$D = U(-baU)^*.$$

Note that (3) implies

$$U(-baU)^* = 1 + (x - ab + aU(-baU)^*b)U(-baU)^*$$

therefore,

$$\begin{aligned} U &= 1 + baU + (x - ab)U + aU(-baU)^*bU \\ &= 1 + (x - ab + ba)U + aU(1 - baU + baUbaU - \dots)bU \\ &= 1 + (x - ab + ba)U + aUbU - aUbaUbaU + aUbaUbaUbaU - \dots \\ &= 1 + (x - ab + ba)U + (-aUb)^*aUbU. \end{aligned}$$

Hence

$$(1 + aUb)U = 1 + aUb + (1 + aUb)(x - ab + ba)U + aUbU$$

and

$$\begin{aligned} U &= 1 + aUb + (1 + aUb)(x - ab + ba)U \\ &= (1 + aUb)(1 + (x - ab + ba)U). \end{aligned}$$

QED

Remark 5. If we put $x = ab - ba$ in the last equation, then $U = 1 + aUb$ which implies $U = \sum_{n \geq 1} a^{n-1}b^{n-1}$ and $1 + aUb = \sum_{n \geq 0} a^n b^n$.

Note that Theorem 4 does not imply that all coefficients in U as series in a, b and x are positive. However, simple computations show that the inversion of the series $1 - aDb$ is written in the form

$$1 + au_1b + au_2b + \dots$$

where the degree of u_n is $2n - 2$, $n \geq 1$ and

$$u_1 = 1,$$

$$u_2 = ba + x,$$

$$u_3 = (ba)^2 + xba + bax + axb + x^2,$$

$$\begin{aligned} u_4 = & (ba)^3 + x(ba)^2 + baxba + (ba)^2x + a^2xb + axb^2a + ba^2xb \\ & + x^2ba + xba x + bax^2 + ax^2b^2 + axbx + xaxb + x^3, \end{aligned}$$

and so on. The positivity follows from the path interpretation at the beginning of the section.

Problem 6: How to write a recurrence relations on u_n similar to relations (1). It must imply that the number of terms for u_n is the n -th Catalan number. It also must show that if $x = ab - ba$ then $u_n = a^{n-1}b^{n-1}$.

We may set $x = 1$ and get

$$u_1 = 1, \quad u_2 = ba + 1, \quad u_3 = (ba)^2 + 2ba + ab + 1,$$

$$u_4 = (ba)^3 + 3(ba)^2 + ab^2 + ba^2b + a^2b^2 + 3ba + 3ab + 1.$$

Problem 7: How to describe polynomials u_n for this and other specializations? Any relations with known polynomials?

3. The Quasideterminant of a Jacobi Matrix

In this section we discuss solutions of noncommutative quadratic equation (2) using quasideterminants. Recall ([1]) that quasideterminant $|A|_{pq}$ of the matrix $A = (a_{ij})$, $i, j = 1, 2, \dots$ is defined as follows. Let A^{pq} be the submatrix of A obtained from A by removing its p -th row and q -th column. Denote by r_p and c_q be the p -th row and the q -th column of A with element a_{pq} removed. Assume that matrix A^{pq} is invertible. Then

$$|A|_{pq} := a_{pq} - r_p(A^{pq})^{-1}c_q.$$

Let now $A = (a_{ij})$, $i, j \geq 1$ be a Jacobi matrix, i.e. $a_{ij} = 0$ if $|i - j| > 1$. Set $T = I - A$, where I is the infinite identity matrix. Recall that

$$|T|_{11}^{-1} = 1 + \sum a_{1j_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_k 1}$$

where the sum is taken over all tuples (j_1, j_2, \dots, j_k) , $j_1, j_2, \dots, j_k \geq 1$, $k \geq 1$.

Also,

$$|T|_{11} = 1 - a_{11} - \sum a_{1j_1} a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_k 1}$$

where the sum is taken over all tuples (j_1, j_2, \dots, j_k) , $j_1, j_2, \dots, j_k > 1$, $k \geq 1$.

Assume that the degree of all diagonal elements a_{ii} is two and the degree of all elements a_{ij} such that $i \neq j$ is one. Then

$$|T|_{11}^{-1} = 1 + \sum_{n \geq 1} t_n \quad (3)$$

where t_n is homogeneous polynomial of degree $2n$ in variables a_{ij} .

In particular,

$$t_1 = a_{11} + a_{12}a_{21},$$

$$t_2 = a_{11}^2 + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11} + a_{12}a_{22}a_{21} + (a_{12}a_{21})^2 + a_{12}a_{23}a_{32}a_{21}.$$

Note that each monomial corresponds, in a one-to-one way, to a ‘‘Schröder walk’’ [2] <http://oeis.org/A006318>, hence:

Proposition 8: The number of monomials of t_n is the n -th Large Schröder Number.

If we set $a_{11} = 0$ we get walks obviously counted by the ‘‘little’’ Schröder numbers [2] <http://oeis.org/A001003>, hence:

Proposition 9: Set $a_{11} = 0$. Then the number of monomials in each t_n is $A001003[n]$.

Let now a, x, b be formal variables, the degree of a and b is one and the degree of x is two. Set $a_{ii} = x - ab$, $a_{i,i+1} = a$, $a_{i+1,i} = b$ for all i . By the definition of quasideterminants, we have

$$|T|_{11} = 1 - x + ab - a|T|_{11}^{-1}b.$$

Denote $|T|_{11}^{-1}$ by D . Then last equation can be written as

$$D^{-1} = 1 - x + ab - aDb$$

or

$$D = 1 + D(x - ab) + DaDb$$

which is exactly our equation (2).

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