WILF'S "SNAKE OIL" METHOD PROVES AN IDENTITY IN THE MOTZKIN TRIANGLE

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Dedicated to the memory of Herb Wilf

Abstract. We give yet-another illustration of using Herb Wilf's *Snake Oil Method*, by proving a certain identity between the entries of the so-called *Motzkin Triangle*, that arose in a recent study of enumeration of certain classes of integer partitions. We also briefly illustrate how this method can be applied to general 'triangles'.

Our starting point was a certain conjecture, concerning the so-called *simultaneous core* partitions, found in a recent preprint [2, Conjecture 11.5]. It reads:

Conjecture. *Let s and d be two coprime positive integers. Then the number of $(s, s+d, s+2d)$-core partitions is given by*

$$\sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s+d-1}{2k+d-1} \binom{2k+d}{k} \frac{1}{2k+d}.$$

We then paid particular interest to the special cases $d = 1$ (see [1]), resulting in the *Motzkin numbers* proven in [2] and by Yang-Zhong-Zhou [5], and $d = 2$ initiating yet another link [1, Problem 11.6] to the Motzkin triangle which we now state.

Problem. The *Motzkin triangle* $T(n,k)$ of numbers is defined according to the rules:
(1) $T(n,0) = 1$;
(2) $T(n,k) = 0$ if $k < 0$ or $k > n$;
(3) $T(n,k) = T(n-1,k-2) + T(n-1,k-1) + T(n-1,k)$.

Prove the identity (this is sequence A026940 in OEIS [3]))

$$\sum_{k=0}^{n} T(n,k)T(n,k+1) = \sum_{k=0}^{n} \binom{2n}{2k+1} \binom{2k+1}{k} \frac{1}{k+2}.$$

Let us first observe that any such identity is nowadays *automatically provable*, thanks to the so-called *Wilf-Zeilberger algorithmic proof theory*, but it is still fun to prove it, whenever possible, the old-fashioned way, by purely *human* means. We will do this, by using what Herb Wilf called the Snake-Oil method [4, Section 4.3].

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Recall that the *Constant Term* of a Laurent polynomial, $P(x)$, is the coefficient of x^0. For example, $CT[4/x+3+5x]=3$.
For motivation, let's look at a few known examples.
(1) $\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n$ and hence $\mathrm{CT}\left[\frac{(1+x)^n}{x^k}\right]=\binom{n}{k}$, the famous binomial coefficients as entries in the familiar Pascal's triangle (see $A007318$ in OEIS [3]).
(2) $\sum_{k=0}^{n+2}C(n,k)\,x^k=(1+x)^n(1-x)$, this is one variant among the Catalan triangles (see the sequences $A008315$ and $A037012$ in OEIS [3]).
(3) $\sum_{k=0}^{2n}t(n,k)\,x^k=(1+x+x^2)^n$, the trinomial triangle (see $A027907$ in OEIS [3]).

Going back to the Motzkin triangle , we return to our Problem by first *extending* the definition of the Motzkin triangle from $k=0,1,\dots,n$ to $k=0,1,\dots,2n+2$ as a skew-symmetric sequence:

$$T(n,k)=-T(n,2n-k+2).$$

Note. $T(n,n+1)=0$ and the extended Motzkin triangle (we continue to denote by $T(n,k)$) obeys the *same* recurrence. As a result, it is easy to construct the generating function

$$\sum_{k=0}^{2n+2}T(n,k)x^k=(1+x+x^2)^n(1-x^2).$$

Or, equivalently, for $k\in\{0,1,\dots,2n+2\}$,

$$T(n,k)=\mathrm{CT}\left(\frac{(1+x+x^2)^n(1-x^2)}{x^k}\right).$$

So, the stage is now set and The Snake Oil method can be brought to bear:

$$\begin{aligned}
\sum_{k=0}^{n}T(n,k)T(n,k+1) &= \frac{1}{2}\sum_{k=0}^{2n+2}T(n,k)T(n,k+1)\\
&= \frac{1}{2}\sum_{k=0}^{2n+1}T(n,k)\,\cdot\mathrm{CT}\left(\frac{(1+x+x^2)^n(1-x^2)}{x^{k+1}}\right)\\
&= \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^n(1-x^2)}{x}\sum_{k=0}^{2n+1}T(n,k)x^{-k}\right]\\
&= \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^n(1-x^2)}{x}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n\left(1-\frac{1}{x^2}\right)\right]\\
&= \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{2n}(1-x^2)}{x^{2n+1}}\right]-\frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{2n}(1-x^2)}{x^{2n+3}}\right]\\
&= \frac{1}{2}\,T(2n,2n+1)-\frac{1}{2}\,T(2n,2n+3)\\
&= \frac{1}{2}\,T(2n,2n-1);
\end{aligned}$$

where the last equality is due to $T(2n,2n+1)=0$ and $T(2n,2n+3)=-T(2n,2n-1)$.
We pause for a moment to appreciate a striking similarity between the two identities,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k+1}=\binom{2n}{n+1}\qquad\text{and}\qquad\sum_{k=0}^{n}T(n,k)\,T(n,k+1)=\frac{1}{2}\,T(2n,2n-1),$$

involving coefficients in the Pascal's triangle and the current Motzkin's triangle, respectively.

On the other hand, if we expand $(1+x+x^2)^{2n} = ((1+x)+x^2)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^{2k}(1+x)^{2n-k}$ then by reverse-engineering the expression for $\frac{1}{2}\,T(2n,2n-1)$ from above, we are led to

$$\begin{aligned}
\frac{1}{2}\,T(2n,2n-1) &= \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{2n}(1-x^2)}{x^{2n-1}}\right] \\
&= \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{2n}}{x^{2n-1}}\right] - \frac{1}{2}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{2n}}{x^{2n-3}}\right] \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\binom{2n}{k}\binom{2n-k}{2n-2k-1} - \frac{1}{2}\sum_{k=0}^{n-2}\binom{2n}{k}\binom{2n-k}{2n-2k-3} \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{2k+1}{k} - \frac{1}{2}\sum_{k=0}^{n-2}\binom{2n}{2k+3}\binom{2k+3}{k} \\
&= n + \frac{1}{2}\sum_{k=1}^{n-1}\binom{2n}{2k+1}\binom{2k+1}{k} - \frac{1}{2}\sum_{k=1}^{n-1}\binom{2n}{2k+1}\binom{2k+1}{k-1} \\
&= n + \sum_{k=1}^{n-1}\binom{2n}{2k+1}\binom{2k+1}{k}\frac{1}{k+2} = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{2k+1}{k}\frac{1}{k+2},
\end{aligned}$$

which is exactly the right-hand side of our problem. This completes the proof. In fact, we have improved the assertion of Problem 11.6 because of our success in evaluating the two sums into the *simpler* form $\frac{1}{2}T(2n,2n-1)$. Therefore, we may formulate our conclusion as the next result.

Theorem 1. *The following identities hold true:*

$$\sum_{k=0}^{n} T(n,k)T(n,k+1) = \sum_{k=0}^{n}\binom{2n}{2k+1}\binom{2k+1}{k}\frac{1}{k+2} = \frac{1}{2}\,T(2n,2n-1).$$

A litmus test (or a cannon measure, if you prefer) to the quality of a good technique is perhaps its enlightenment, simplicity and implications. Indeed, in our case, the linear operator CT offers both a clue to and a proof for an *effortless* generalization of Theorem 1. The Motzkin triangle persists!

Theorem 2. *The following identity holds true:*

$$\sum_{k=0}^{\lfloor \frac{s}{2}\rfloor}\binom{s+d-1}{2k+d-1}\binom{2k+d-1}{k}\frac{1}{k+d} = \frac{1}{d}\,T(s+d-1,s).$$

Proof. This is completely analogous to what has been demonstrated earlier. To wit,

$$\begin{aligned}
\frac{1}{d}\,T(s+d-1,s) &= \frac{1}{d}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{s+d-1}}{x^s}\right] - \frac{1}{d}\,\mathrm{CT}\left[\frac{(1+x+x^2)^{s+d-1}}{x^{s-2}}\right] \\
&= \frac{1}{d}\sum_{k\geq 0}\binom{s+d-1}{k}\binom{s+d-k-1}{s-2k} - \frac{1}{d}\sum_{k\geq 0}\binom{s+d-1}{k}\binom{s+d-k-1}{s-2k-2} \\
&= \frac{1}{d}\sum_{k\geq 0}\binom{s+d-1}{2k+d-1}\binom{2k+d-1}{k} - \frac{1}{d}\sum_{k\geq 0}\binom{s+d-1}{2k+d+1}\binom{2k+d+1}{k} \\
&= \sum_{k\geq 0}\binom{s+d-1}{2k+d-1}\binom{2k+d-1}{k}\frac{1}{k+d}.
\end{aligned}$$

The proof is complete. $\square$

Is there more? Yes, here is a *bonus*! As a nice implication of the preceding results, the above Conjecture may be stated much more succinctly.

Conjecture. *If $s, d \geq 1$ are coprime integers, then the number of $(s, s+d, s+2d)$-core partitions equals*

$$\frac{1}{d}\, T(s+d-1, s).$$

General Triangles. The above method of proof extends to a much wider class of triangle of numbers generated by the family

$$\{P(x)^n Q(x) : n \in \mathbb{N}\}$$

where the polynomial $P(x)$ is palindromic. For the sake of simplicity we will take $Q(x) = 1 - x^2$. Consider for instance the sequence $A(n,k)$ defined by the recurrence

$$A(n,k) = a_0 A(n-1,k) + a_1 A(n-1,k-1) + \cdots + a_d A(n-1,k-d)$$

satisfying some initial conditions and where $a_j = a_{d-j}$ for $j \in \{0, 1, \ldots, d\}$ (palindromic coefficients). As before, extend the definition of $A(n,k)$ as skew-symmetric. If we take $P(x) = \sum_{j=0}^{d} a_j x^j$ and $Q(x) = 1 - x^2$ then

$$\sum_{k \geq 0} A(n,k)\, x^k = P(x)^n Q(x).$$

Once more, The Snake Oil method delivers the argument almost verbatim:

$$\begin{aligned}
\sum_{k=0}^{dn/2} A(n,k)A(n,k+1) &= \frac{1}{2} \sum_{k=0}^{dn+2} A(n,k)A(n,k+1) \\
&= \frac{1}{2} \sum_{k=0}^{dn+2} A(n,k) \cdot \mathrm{CT}\left(\frac{P(x)^n(1-x^2)}{x^{k+1}}\right) \\
&= \frac{1}{2}\, \mathrm{CT}\left[\frac{P(x)^n(1-x^2)}{x} \sum_{k=0}^{dn+2} A(n,k) x^{-k}\right] \\
&= \frac{1}{2}\, \mathrm{CT}\left[\frac{P(x)^n(1-x^2)}{x} P\left(1/x\right)^n \left(1 - \frac{1}{x^2}\right)\right] \\
&= \frac{1}{2}\, \mathrm{CT}\left[\frac{P(x)^{2n}(1-x^2)}{x^{dn+1}} \left(1 - \frac{1}{x^2}\right)\right] \\
&= \frac{1}{2}\, \mathrm{CT}\left[\frac{P(x)^{2n}(1-x^2)}{x^{dn+1}}\right] - \frac{1}{2}\, \mathrm{CT}\left[\frac{P(x)^{2n}(1-x^2)}{x^{dn+3}}\right] \\
&= \frac{1}{2} A(2n, dn+1) - \frac{1}{2} A(2n, dn+3) \\
&= \frac{1}{2} A(2n, dn-1);
\end{aligned}$$

where the last equality is due to $A(2n, dn+1) = 0$ and $A(2n, dn+3) = -A(2n, dn-1)$.

References

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