TOWARDS A WZ EVALUATION OF THE MEHTA INTEGRAL

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Dedicated with affection and admiration to Dick Askey, on his 60^{th} *Birthday.*

Dick Askey [As] proposed the problem of proving the Mehta (see [M]) integral identity:

$$\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} exp(-x_1^2/2 - \dots - x_n^2/2) \prod_{1<=i<j<=n} (x_i - x_j)^{2c} dx_1 \dots dx_n = \prod_{j=1}^{n} \frac{(cj)!}{j!} \ , \quad (Mehta)$$

without using Selberg's integral (see [M]). This problem was solved by Greg Anderson[An]. Here we use the method of [WZ1][WZ2] to initiate another Selberg-free proof, that we believe is of independent interest. We show that (Mehta) for any given n, is equivalent to the following elegant identity $(d := n(n-1)/2)$:

$$\{\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1<=i<j<=n} \frac{1}{x_i - x_j} (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\}^d \prod_{1<=i<j<=n} (x_i - x_j)^2 \ = \ 2^d d! n! \prod_{1<=s<r<=n} (rc + s) \qquad (Mehta')$$

$(Mehta')$ is purely routine for any specific n, but at the time of writing we are unable to prove it directly for general n. Of course, we do have *a* proof, since we are going to show that (Mehta') and (Mehta) are equivalent, but what we are after is a *direct* proof. The author is offering a prize of 25 US dollars for such a proof.

The present method also extends obviously to the Macdonald-Mehta integral[M], that was proved by Beckner and Regev[BR] for the classical root systems (see [M]), by Garvan[G] for the exceptional root system F_4, and by Opdam[O] for E_6, E_7, and E_8. It follows that the present approach should also yield new proofs for all the exceptional root systems, at least in principle, but most likely also in practice. More important, it seems to have a high chance of producing a uniform, intrinsic, classification-independent proof. We leave to the reader, as an instructive exercise, the task of finding the root-system analog of (Mehta') that is equivalent to the now-proved Mehta-Macdonald conjecture, and we offer an additional 25 dollars for an intrinsic proof.

Our proposed proof of (Mehta) will be a *derivation* rather than a *verification*, and will follow the method of [WZ2]. Let's call the left of (Mehta) $L(c)$, and let's call the integrand $F(c; x_1, \dots, x_n)$. We know from the general theory of [WZ2] that for some r and some rational functions $P_1, \dots, P_n$, in $(c, x_1, \dots, x_n)$, and some rational functions in c, $a_0(c), \dots, a_r(c)$,

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$$\sum_{s=0}^{r} a_s(c)F(c+s;x_1,...,x_n) \;=\; \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(P_iF) \ . \qquad (WZ)$$

Let's be optimistic and try out $r=1$. Without loss of generality, set $a_1 := 1$. Substituting F in (WZ), performing all differentiations, and dividing throughout by F, leads to the following equation for the P_i and a_0 :

$$a_0 + \prod_{1<=i<j<=n} (x_i - x_j)^2 = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} + 2c\sum_{i=1}^{n}(\sum_{j\neq i} \frac{1}{x_i - x_j})P_i - \sum_{i=1}^{n} x_iP_i \ . \qquad (1)$$

Let's be even more optimistic, and assume that the P_i are polynomials, rather than mere rational functions, in their dependence on $(x_1,...,x_n)$, and are further the components of the gradient of another polynomial P, i.e. $P_i = \partial P/\partial x_i$, $i=1,...,n$, for some polynomial P. Equation (1) then becomes

$$a_0 + \prod_{1<=i<j<=n} (x_i - x_j)^2 = \{\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i=1}^{n}(\sum_{j\neq i} \frac{1}{x_i - x_j})\frac{\partial}{\partial x_i}\}\, P \ - \sum_{i=1}^{n}(x_i\frac{\partial}{\partial x_i})P \ . \qquad (2)$$

If we can find such a polynomial P, and compute the corresponding a_0, then it would follow from (WZ), upon integrating w.r.t $x_1,...,x_n$, that $L(c)$ satisfies the recurrence $L(c+1) = -a_0(c)L(c)$, which combined with $L(0)=1$, would enable one to find $L(c)$. Note that the mere existence of a_0, that we will shortly prove is given by the left side of (Mehta'), implies that $L(c)$ is of *closed form*, which from a theoretical point of view is almost as good as knowing what it is exactly.

Let's write P as a sum of its homogeneous parts

$$P = \sum_{j=2}^{2d} P^{(j)}, \quad P^{(j)}\ homog.\, of\ deg.\, j$$

where, as above d equals $n(n-1)/2$. Denote the operator inside the braces of (2) (or (Mehta')) by $\mathbf{Z}$. Using Euler's formula, we get

$$a_0 + \prod_{1<=i<j<=n} (x_i - x_j)^2 \;=\; \sum_{j=2}^{2d} \mathbf{Z}P^{(j)} - \sum_{j=2}^{2d} jP^{(j)} \;=\; (-2d)P^{(2d)} + \sum_{j=2}^{2d}(\mathbf{Z}P^{(j)} - (j-2)P^{(j-2)}) \ . \qquad (3)$$

Equating corresponding homogeneous parts, we get

$$P^{(2d)} = -(2d)^{-1} \prod_{1<=i<j<=n} (x_i - x_j)^2 \ , \quad P^{(j-2)} \;=\; \frac{1}{j-2}\mathbf{Z}P^{(j)} \ , \ j = 2d\, ,\ 2d-2\, ,\ ...,4\ . \qquad (4)$$

It is easy to see that $\mathbf{Z}$ maps homogeneous symmetric polynomials to homogeneous symmetric polynomials, and that it reduces the degree by 2. Iterating (4), and comparing the constant part of (3) finally yields that both P and a_0 indeed exist, and that

$$a_0 = -(2^d d!)^{-1} \mathbf{Z}^d [\prod_{1<=i<j<=n} (x_i - x_j)^2] \ .$$

Hence Mehta's integral is indeed expressible in closed form, and proving that its value coincides with the value implied by (Mehta) amounts to proving (Mehta'). QED

The referee noticed empirically the following generalization. For a, any positive integer, we have:

$$\{ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1<=i<j<=n} \frac{1}{x_i - x_j} (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}) \}^{ad} \prod_{1<=i<j<=n} (x_i - x_j)^{2a} = \qquad (Mehta'')$$

$$2^{ad} (da)! n! a \prod_{2<=r<=n} \prod_{1<=s<=ra \, , \, s \neq 0 \ mod \ r} (rc + s) \ .$$

It turns out that (Mehta") follows from (Mehta) exactly the same way as (Mehta'), just replace c by ca, in (Mehta), and repeat the argument! In particular, it follows that (Mehta') implies (Mehta") albeit, indirectly, via (Mehta).

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