A Short Proof of McDougall's Circle Theorem

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Abstract. This note offers a short, elementary proof of a result similar to Ptolemy's theorem. Specifically, let $d_{i,j}$ denote the distance between P_i and P_j. Let n be a positive integer and P_i, for $1 \le i \le 2n$, be cyclically ordered points on a circle. If

$$R_i := \prod_{\substack{1 \le j \le 2n \\ j \ne i}} d_{i,j},$$

then

$$\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}. \tag{1}$$

In fond memory of Andrei Zelevinsky (1953–2013),
who loved Ptolemy's Theorem.

Ptolemy's Theorem is a beautiful, classical result concerning quadrilaterals. Specifically, let P_1, P_2, P_3, and P_4 be cyclically ordered points on a circle and $d_{i,j}$ denote the distance between P_i and P_j. Then Ptolemy's Theorem states that $d_{1,3}d_{2,4} = d_{1,2}d_{3,4} + d_{1,4}d_{2,3}$. Refinements, known as the Brahmagupta–Mahavira identities [**1**], lead to a "ratio version" of Ptolemy's Theorem:

$$\frac{d_{1,3}}{d_{2,4}} = \frac{d_{1,2}\, d_{1,4} + d_{2,3}\, d_{3,4}}{d_{1,4}\, d_{3,4} + d_{1,2}\, d_{2,3}}. \tag{2}$$

Equation (2) can be written as

$$\frac{1}{d_{1,2}\, d_{1,3}\, d_{1,4}} + \frac{1}{d_{3,1}\, d_{3,2}\, d_{3,4}} = \frac{1}{d_{2,1}\, d_{2,3}\, d_{2,4}} + \frac{1}{d_{4,1}\, d_{4,2}\, d_{4,3}}.$$

Jane McDougall [**2**] has generalized this result from 4 points to $2n$ points.

Theorem 1.1 (McDougall). *Let n be a positive integer and P_i, for $1 \le i \le 2n$, be cyclically ordered points on a circle. If*

$$R_i := \prod_{\substack{1 \le j \le 2n \\ j \ne i}} d_{i,j},$$

then

$$\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}. \tag{3}$$

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McDougall's proof, using tools from complex analysis, follows by applying harmonic mappings to a class of minimal surfaces. Related use of these methods can be found in [**3**]. The goal of this note is to provide a short, elementary proof of McDougall's theorem. The key to the proof is using the *Lagrange Interpolation Formula*: If $P(z)$ is a polynomial whose degree does not exceed $N-1$, then

$$P(z) = \sum_{i=1}^{N} \frac{(z-z_1)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_N)}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)} P(z_i)$$

for any distinct numbers $z_1, \ldots, z_N$. Using a formula like this is not surprising, since Equation (3) involves sums of products.

Proof. Without loss of generality, assume that the circle is centered at the origin and has radius one. Denote the points as $P_i = (\cos 2t_i, \sin 2t_i)$, for $1 \le i \le 2n$, where $0 \le t_1 < t_2 < \cdots < t_{2n} < \pi$. Basic trigonometry can be used to show that $d_{i,j} = 2\sin(t_i - t_j)$. Letting $u_i = e^{It_i}$ where $I = \sqrt{-1}$, it follows that

$$d_{i,j} = -I\left(\frac{u_i^2 - u_j^2}{u_i u_j}\right)$$

whenever $i < j$. This produces

$$I^{2n-1}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right) \frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

With Equation (3) in view, we construct

$$I^{2n-1}\sum_{i=1}^{2n}(-1)^{i-1}\frac{1}{R_i} = \left(\prod_{j=1}^{2n} u_j\right) \sum_{i=1}^{2n} \frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}. \tag{4}$$

Proving the theorem is therefore equivalent to proving that the sum on the right side of Equation (4) equals zero.

Applying the Lagrange Interpolation Formula with $P(z) = z^r$, where $r < N-1$, the coefficient of z^{N-1} yields

$$0 = \sum_{i=1}^{N} \frac{z_i^r}{(z_i-z_1)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_N)}. \tag{5}$$

Taking $N = 2n$, $r = n-1$ and $z_i = u_i^2$ for $i = 1, 2, \ldots, 2n$, the z_i terms are distinct and Equation (5) becomes

$$0 = \sum_{i=1}^{2n} \frac{u_i^{2n-2}}{\prod_{j\neq i}(u_i^2 - u_j^2)}.$$

This proves the theorem. ∎

Remark 1. The Lagrange Interpolation Formula is easy to prove; the left and right sides agree when $z = z_i$, for $1 \leq i \leq N$; thus, the two degree $N - 1$ polynomials must agree everywhere. Therefore, no sophisticated machinery is needed to prove McDougall's Theorem.

Remark 2. Letting the circle's radius approach infinity gives the corollary that Equation (3) holds if the $2n$ points are collinear. In fact, the equation still holds on lines even if the number of points is odd:

$$\sum_{i=1}^{2n-1} \frac{(-1)^i}{R_i} = 0.$$

We leave this as an exercise for the reader.

REFERENCES

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New societies, almost inevitably, lead to the setting-up of new journals. In the case of the Mathematical Association of America, almost the reverse is true! In mathematics, prior to 1850, there had been short-lived largely problem solving journals. There were also problem columns or sections in such general journals as the *Saturday Evening Post*. However, the mathematical journals continuing from the nineteenth century to the present are the *American Journal of Mathematics* (1878), *Annals of Mathematics* (1884), *Bulletin of the New York Mathematical Society* (1891) (now the *Bulletin of the A.M.S.*), and *The American Mathematical Monthly* (1894). The first three are chiefly concerned with pure mathematical research. The story of the founding of the Association and of the support of collegiate mathematical instruction is largely involved with the fourth, the MONTHLY.

The Mathematical Association of America:
It's First Fifty Years.
Edited by Kenneth O. May,
Mathematical Association of America,
Washington DC, 1972, p. 18.