On Invariance Properties of Entries of Matrix Powers

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Abstract: A few years ago, Peter Larcombe discovered an amazing property regarding two by two matrices. For any such 2×2 matrix A, the ratios of the two anti-diagonal entries is the same for all powers of A. We discuss extensions to higher dimensions, and give a short bijective proof of Larcombe and Eric Fennessey's elegant extension to tri-diagonal matrices of *arbitrary* dimension. This article is accompanied by a Maple package.

Peter Larcombe's Surprising Discovery

In [2], Peter Larcombe gave four proofs of a seemingly new and amazing property of a 2×2 matrix, for any such matrix (we denote the (i, j) entry of a matrix B by B_{ij})

$$A_{12} \cdot (A^m)_{21} = A_{21} \cdot (A^m)_{12} \quad ,$$

for *all* positive integers m.

We first observe that, in hindsight (but only in hindsight!) this is not that surprising. More generally, for a general $n \times n$ matrix A, and any subset S of cardinality $n+1$ of the set of n^2 entries $\{(i, j) \,|\, 1 \leq i, j \leq n\}$, there exist polynomials $q_s(A)$ in the entries of A (independent of m) such that

$$\sum_{s \in S} q_s \cdot (A^m)_s \;=\; 0 \quad , \tag{1}$$

for all $m > 1$.

This fact follows from the **Cayley-Hamilton** equation that says that If $P_A(x) := \det(A - x\, I)$ is the **characteristic polynomial** of A, then the $n \times n$ matrix $P_A(A)$ equals the **zero matrix** $\mathbf{0}$. Writing

$$P_A(x) = \sum_{k=0}^{n} p_k x^k \quad ,$$

we have

$$\sum_{k=0}^{n} p_k A^k = \mathbf{0} \quad ,$$

where $\mathbf{0}$ is the all-zero matrix. Multiplying by A^m we get

$$\sum_{k=0}^{n} p_k A^{m+k} = \mathbf{0} \quad ,$$

for all m.

Taking the ij entry, we have that each of the n^2 sequences $(A^m)_{ij}$ satisfy the **same** n^{th}-order linear recurrence equation with **constant coefficients**

$$\sum_{k=0}^{n} p_k (A^{m+k})_{ij} = 0 \quad .$$

It is well-known and easy to see ([1][8]) that any $n+1$ sequences that satisfy the **same** recurrence of order n, must be **linearly dependent**. Also, in order to find the relation, it is enough to find $n+1$ values of the q_s for which (1) holds for the $m = 1, 2, \ldots, n$. Then this linear combination also satisfies that very same linear recurrence, and since it **vanishes** at the first n **initial conditions** it must be **identically zero**.

If our subset S of entries only consists of non-diagonal entries, we can do even better, Eq. (1) is true for any set S of n non-diagonal entries.

Note that the Cayley-Hamilton equation implies that

$$\sum_{k=1}^{n} p_k A^k$$

is a **diagonal** matrix (namely $-\det(A)\mathbf{I}$), hence for a non-diagonal entry ij ($i \neq j$), $(A^m)_{ij}$ always satisfies the **same** linear recurrence equation (with constant coefficients) of order $n-1$, hence any n such non-diagonal entries must be **linearly dependent**. In the original case of a 2×2 matrix discussed in [2], it follows that the $(1,2)$ and $(2,1)$ entries of A^m always satisfy the **same** relation as those of A, hence we have yet-another-proof (without equations!) of Larcombe's amazing discovery.

But what about higher dimensions? Now things get much more complicated, and we need a computer algebra system (in our case Maple). For example, we have the following

Theorem: Let $A = (a_{ij})_{1 \leq i,j \leq 3}$ be a 3×3 matrix, then for **all** $m \geq 1$, we have

$$(a_{12}a_{21}a_{23} - a_{13}a_{21}a_{22} + a_{13}a_{21}a_{33} - a_{13}a_{23}a_{31}) \cdot (A^m)_{12}$$

$$+ (a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}) \cdot (A^m)_{13}$$

$$+ \left(-a_{12}^2 a_{23} + a_{12}a_{13}a_{22} - a_{12}a_{13}a_{33} + a_{13}^2 a_{32}\right) \cdot (A^m)_{21} \; = \; 0 \quad .$$

There are three more such theorems (up to trivial isomorphism) for the $n = 3$ case, while there are 27 inequivalent cases for $n = 4$. They can all be found in the following output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oLarcombe1.txt` .

We were unable to find the corresponding relations for $n = 5$, they got too complicated!

A bijective proof of the Larcombe-Fennessey theorem about Tridiagonal matrices

Any matrix identity for a fixed dimension is essentially *high-school algebra* and can be verified by a computer algebra system, even if the power m is arbitrary. But the following theorem of Larcombe and Fennessey, regarding **tridiagonal** matrices of *arbitrary* dimension is *university algebra* and is more interesting.

Theorem (Larcombe and Fennessey [3][6]) : Let A be a general $n \times n$ tridiagonal matrix (for *any* $n \geq 2$), then for all $1 \leq i < n$, and all $m \geq 1$, we have

$$a_{i,i+1} \cdot (A^m)_{i+1,i} \;=\; a_{i+1,i} \cdot (A^m)_{i,i+1} \quad . \tag{2}$$

For a concrete example see [3].

We will give a **combinatorial proof**. Fix n and let $a_{i,j}$ $(1 \leq i, j \leq n)$ be n^2 **commuting** indeterminates. It follows immediately from the definition of matrix multiplication that the (i,j) entry of A^m is the **weight-enumerator** of the set of $(m+1)$-letter words in the alphabet $\{1, 2, \ldots, n\}$ whose first letter is i and last letter is j, with the weight

$$Weight(w_1 \ldots w_{m+1}) := a_{w_1\, w_2} \; \cdot \; a_{w_2\, w_3} \cdots \; a_{w_{m-1}\, w_m} \; \cdot \; a_{w_m\, w_{m+1}} \quad .$$

For example $Weight(123123) = a_{12}a_{23}a_{31}a_{12}a_{23}$.

If our matrix is *tridiagonal*, then all the words are *continuous* i.e. after the letter i can only come one of the (up to) three letters $\{i-1, i, i+1\}$. For example if $n = 5$ then the following is a legal word

$$2333234333221122112234455443 \quad ,$$

but the following one is **not**

$$233312 \quad ,$$

because after the fourth letter, that is a '3', comes the letter '1'.

Let $\mathcal{W}_m(i,j)$ be the set of legal $(m+1)$-letter words in the alphabet $\{1, 2, \ldots, n\}$ that start with the letter i and end with the letter j. Its weight-enumerator (i.e. sum of the weights of its members) is $(A^m)_{ij}$, where now A is a generic $n \times n$ **tridiagonal** matrix. For ease of type-setting let $i' := i+1$.

Note that:

• The left side of (2) is the weight-enumerator of the set of words, $i\,\mathcal{W}_m(i', i)$, which is the set of **legal** $(m+2)$-letter words that start and end with the letter i, and whose **second** letter is i'.

• The right side of (2) is is the weight-enumerator of $i'\,\mathcal{W}_m(i, i')$ which is the set of **legal** $(m+2)$-letter words that start and end with the letter i', and whose **second** letter if i.

We claim that The mapping

$$T_i \,:\, i\,\mathcal{W}_m(i',i) \rightarrow i'\,\mathcal{W}_m(i,i') \quad ,$$

to be defined next, is a **weight-preserving** bijection.

Let $w = w_1 w_2 \dots w_{m+2}$ be a member of $i\,\mathcal{W}_m(i',i)$, then of course $w_1 = i$ and $w_2 = i'$, and $w_{m+2} = i$. Let k be the **smallest** index such that $w_k = i', w_{k+1} = i$. Of course it exists (by "*continuity*").

Case I: $k = 2$.

If $m = 1$ then the word must be $i\,i'i$ and we map it to $i'\,i\,i'$.

Otherwise we can write

$$w = i\,i'\,i\,u\,i \quad ,$$

for some $(m-2)$-letter word u, and we define

$$T_i(w) := i'\,i\,u\,i\,i' \quad ,$$

that of course belongs to $i'\,\mathcal{W}_m(i,i')$.

Case II: $k = m+1$, then we can write

$$w = i\,i'\,u\,i'\,i \quad ,$$

for some $(m-2)$-letter word u, and we map it to the

$$T_i(w) := i'\,i\,i'\,u\,i' \quad ,$$

that of course belongs to $i'\,\mathcal{W}_m(i,i')$.

Case III: $2 < k < m+1$. Then we can write

$$w = i\,i'\,u\,i'\,i\,v\,i \quad ,$$

for some words u and v, whose total length is $m-2$, and we define

$$T_i(w) := i'\,i\,v\,i\,i'\,u\,i' \quad ,$$

that of course belongs to $i' \, \mathcal{W}_m(i,i')$.

Let's state the inverse mapping

$$U_i \, : \, i' \, \mathcal{W}_m(i,i') \rightarrow i \, \mathcal{W}_m(i',i) \quad .$$

Let $w = w_1 w_2 \dots w_{m+2}$ be a member of $i' \, \mathcal{W}_m(i,i')$, then of course $w_1 = i'$ and $w_2 = i$, and $w_{m+2} = i'$. Let k be the **largest** index such that $w_k = i, w_{k+1} = i'$. Of course it exists (by "*continuity*").

Case I: $k = m+1$. If $m = 2$ the $w = i' \, i \, i'$ and we let $U_i(w)$ be $i \, i' \, i$. Otherwise we can write

$$w \, = \, i' \, i \, u \, i \, i' \quad ,$$

for some $(m-2)$-letter word u, and we define

$$U_i(w) := \, i \, i' \, i \, u \, i \quad .$$

Case II: $k = 2$. We can write

$$w \, = \, i' \, i \, i' \, u \, i'$$

and we define

$$U_i(w) := \, i \, i' \, u \, i' \, i \quad .$$

Case III: $2 < k < m+1$. We can write

$$w \, = \, i' \, i \, v \, i \, i' \, u \, i' \quad ,$$

for some words u and v whose lengths add-up to $m-2$, and we define

$$U_i(w) := \, i \, i' \, u \, i' \, i \, v \, i \quad .$$

Readers are welcome to play with the Maple package

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/Larcombe.txt` , that contains procedure `REL` to discover generalized Larcombe relations (mentioned above) and also implements the above bijection (procedures `Ti` and `Ui`, and `CheckTi` verifies it empirically).

In order to use the Maple package, one should have Maple, of course. Then start a Maple session, and type `read 'Larcombe.txt'`. For on-line help, type

`ezra();` .

Darij Grinberg's Extension

Darij Grinberg discovered that our argument proves a bit more. Here is what he wrote to us:

I'd like to remark that (2) holds not only if A is tridiagonal, but more generally if A has the property that

(*) $a_{u,v} = a_{v,u} = 0$ whenever $u \leq i$ and $v > i$ satisfy $v - u > 1$.

(That is, there is a "tridiagonal bottleneck" between i and $i+1$ in A.) Your proof still applies to this generalization, except that the words should not be "continuous" but rather need to pass the $(i, i+1)$ checkpoint whenever they cross the border between "$\leq i$" and "$> i$". Instead of continuity, you thus need to make a "what goes up must come down" argument when constructing the bijection.

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