Proof of Kyle Petersen's Amazing Conjecture Relating the q- and Fibonacci Analogs of n!

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Let the *darga* of a polynomial be the sum of its low-degree and (high) degree, for example,

$$darga(2+q+q^2-q^3) = 0+3 = 3 \quad , \quad darga(q^2+q^3) = 2+3 = 5 \quad , \quad darga(q^3) = 3+3 = 6 \quad .$$

It is immediate that every *symmetric* polynomial of darga d can be written, uniquely, as a linear combination of

$$(1+q)^d \quad , \quad q(1+q)^{d-2} \quad , \quad q^2(1+q)^{d-4} \quad , \ldots$$

in symbols

$$P(q) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \, q^i (1+q)^{d-2i} \quad ,$$

and we will call the *Kyle Transform*, $\hat{P}(z)$, the polynomial, in z,

$$\hat{P}(z) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \, z^i \quad .$$

Kyle Petersen([P]) made the following elegant conjecture, that I will shortly prove.

Fact: Let $A_n(z)$ be the Kyle transform of

$$[n]! := \prod_{i=1}^{n} \frac{1-q^i}{1-q} \quad ,$$

then

$$A_n(-1) = \prod_{i=1}^{n} F_i \quad ,$$

where the F_i are the famous *Fibonacci numbers* defined by $F_0 = 0, F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$.

Proof. For a general symmetric $P(q)$ we have

$$P(q) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \, q^i (1+q)^{d-2i} = (1+q)^d \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \, (\frac{q}{(1+q)^2})^i \quad .$$

Hence

$$\frac{P(q)}{(1+q)^d} = \hat{P}(\frac{q}{(1+q)^2}) \quad .$$

Letting $z = \frac{q}{(1+q)^2}$, we get the quadratic equation

$$zq^2 + (2z-1)q + z = 0 \quad ,$$

whose (unique formal-power series) solution is

$$q = \frac{1-2z-\sqrt{1-4z}}{2z} = C(z) - 1 = zC(z)^2 \quad ,$$

where

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} \quad ,$$

is the unique formal power series solution of $C(z) = 1 + zC(z)^2$ of Catalan fame. Hence

$$\hat{P}(z) = \frac{P(zC(z)^2)}{(1+zC(z)^2)^d} = \frac{P(zC(z)^2)}{C(z)^d} \quad .$$

Plugging-in $z = -1$ we get

$$\hat{P}(-1) = \frac{P(-C(-1)^2)}{C(-1)^d} \quad .$$

Now

$$C(-1) = \frac{1-\sqrt{5}}{-2} = \frac{\sqrt{5}-1}{2} = \phi^{-1} \quad ,$$

where $\phi = (\sqrt{5}+1)/2$ is the famous *golden ratio*. Hence

$$\hat{P}(-1) = \frac{P(-\phi^{-2})}{\phi^{-d}} = \phi^d \, P(-\phi^{-2}) \quad .$$

Applying this to $P(q) = [n]!$, we get

$$A_n(-1) = \phi^{n(n-1)/2} \prod_{i=1}^{n} \frac{1-(-\phi^{-2})^i}{1+\phi^{-2}} = \prod_{i=1}^{n} \frac{\phi^i - (-\phi^{-1})^i}{\phi+\phi^{-1}} = \prod_{i=1}^{n} F_i \quad . \quad \square$$

Reference

[P] Kyle Petersen, *Private communication* (via Richard P. Stanley) during the banquet in honor of Ira Gessel, Brandeis University, May 8, 2015.

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