An Explicit Formula for the Number of Solutions of $X^2 = 0$ in Triangular Matrices Over a Finite Field

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Abstract: We prove an explicit formula for the number of $n \times n$ upper triangular matrices, over $GF(q)$, whose square is the zero matrix. This formula was recently conjectured by Sasha Kirillov and Anna Melnikov[KM].

Theorem: The number of $n \times n$ upper-triangular matrices over $GF(q)$ (the finite field with q elements), whose square is the zero matrix, is given by the polynomial $C_n(q)$, where,

$$C_{2n}(q) = \sum_j \left[\binom{2n}{n-3j} - \binom{2n}{n-3j-1} \right] \cdot q^{n^2-3j^2-j} \quad ,$$

$$C_{2n+1}(q) = \sum_j \left[\binom{2n+1}{n-3j} - \binom{2n+1}{n-3j-1} \right] \cdot q^{n^2+n-3j^2-2j} \quad .$$

Proof: In [K] it was shown that the quantity of interest is given by the polynomial $A_n(q) = \sum_{r \geq 0} A_n^r(q)$, where the polynomials $A_n^r(q)$ are defined recursively by:

$$A_{n+1}^{r+1}(q) = q^{r+1} \cdot A_n^{r+1}(q) + (q^{n-r} - q^r) \cdot A_n^r(q) \quad ; \quad A_{n+1}^0(q) = 1 \quad . \qquad (Sasha)$$

For any Laurent formal power series $P(w)$, let $CT_w P(w)$ denote the coefficient of w^0. Recall that the *q-binomial coefficients* are defined by

$$\binom{m}{n}_q := \frac{(1-q^m)(1-q^{m-1}) \cdots (1-q^{m-n+1})}{(1-q)(1-q^2)\cdots(1-q^n)} \quad , \qquad (Carl)$$

whenever $0 \leq n \leq m$, and 0 otherwise.

The following lemma gives an explicit expression for $A_n^r(q)$.

Lemma 1:

$$A_n^r(q) = CT_w \left[\frac{(1-w)(1+w)^n q^{r(n-r)}}{w^r} \sum_{i=0}^{\infty} (-1)^i q^{-(i+1)i/2 - i(n-2r)} \binom{i+n-2r}{i}_q w^i \right] \quad . \qquad (Anna)$$

Proof: Call the right side of Eq. $(Anna)$, $S_n^r(q)$. Since $S_{n+1}^0(q) = 1$, the lemma would follow by induction if we could show that

$$S_{n+1}^{r+1}(q) \; - \; q^{r+1} \cdot S_n^{r+1}(q) \; - \; (q^{n-r} - q^r) \cdot S_n^r(q) = 0 \quad . \qquad (Sasha')$$

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Supported in part by the NSF. Nov. 28, 1995.

Using the linearity of CT_w, manipulating the series, using the definition $(Carl)$ of the $q-$binomial coefficients, and simplifying, brings the left side of $(Sasha')$ to be $CT_w \Phi_n^r(q,w)$, where Φ_n^r is zero except when n is odd and $r=(n-1)/2$, in which case it is a monomial in q times $\frac{(1-w)(1+w)^n}{w^{r+1}}$, and applying CT_w kills it all the same, thanks to the symmetry of the Chu-Pascal triangle. $\square$

Summing the expression proved for $A_n^r(q)$, yields that

$$A_n(q) = CT_w \left[(1-w)(1+w)^n \cdot \sum_{r=0}^{\infty} \sum_{i=0}^{r} (-1)^i q^{r(n-r)-(i+1)i/2-i(n-2r)} \binom{i+n-2r}{n-2r}_q w^{i-r} \right] \quad .$$

Letting $l=r-i$, and changing the order of summation, yields

$$A_n(q) = CT_w \left[(1-w)(1+w)^n \cdot \sum_{l=0}^{\lfloor n/2 \rfloor} w^{-l} \cdot q^{ln-l^2} \sum_{i=0}^{\lfloor (n-2l)/2 \rfloor} (-1)^i q^{i(i-1)/2} \binom{n-2l-i}{i}_q \right] \quad . \tag{$SumAnna$}$$

Luckily, the inner sum evaluates nicely thanks to:

Lemma 2:

$$\sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i q^{i(i-1)/2} \binom{m-i}{i}_q = (-1)^{\lfloor m/3 \rfloor} q^{m(m-1)/6} \cdot \chi(m \not\equiv 2 \bmod 3) \quad .$$

Proof: While this is unlikely to be new[2], it is also irrelevant whether or not it is new, since this is *now* routine, thanks to the package `qEKHAD`, accompanying [PWZ]. Let's call the left side divided by $q^{m(m-1)/6}$, $Z(m)$. Then we have to prove that $Z_0(m) := Z(3m)$ equals $(-1)^m$, $Z_1(m) := Z(3m+1)$ equals $(-1)^m$, and $Z_2(m) := Z(3m+2)$ equals 0. It is directly verified that these are true for $m=0,1$, and the general result follows from the second order recurrences produced by `qEKHAD`. The input files `inZ0`, `inZ1`, `inZ2` as well as the corresponding output files, `outZ0`, `outZ1`, `outZ2` can be obtained by anonymous `ftp` to `ftp.math.temple.edu`, directory `pub/ekhad/sasha`. The package `qEKHAD` can be downloaded from `http://www.math.temple.edu/~zeilberg`. $\square$

To complete the proof of the theorem, we use lemma 2 to evaluate the inner sum of $(SumAnna)$, then to get $A_{2n}(q)$, we replace n by $2n$, and then replace l by $l+n$, and finally use the binomial theorem. Similarly for $A_{2n+1}(q)$. $\square$

References

[K] A.A. Kirillov, *On the number of solutions to the equation $X^2 = 0$ in triangular matrices over a finite field*, Funct. Anal. and Appl. **29** (1995), no. 1.

[KM] A.A. Kirillov and A. Melnikov, *On a remarkable sequence of polynomials*, preprint.

[PWZ] M. Petkovsek, H.S. Wilf, and D. Zeilberger, *"A=B"*, A.K.Peters, 1996.

[2] For a 'classical' proof, see Christian Krattenthaler's message, at `ftp://ftp.math.temple.edu/pub/ekhad/sasha`.