The multivariable Almqvist-Zeilberger algorithm [?, ?] takes as input an integrand that is hyperexponential in the integration variables and hypergeometric in the discrete parameter, and then computes a single creative telescoping relation [?] that yields a recurrence satisfied by the integral. Alternatively, it can be more efficient to perform creative telescoping on each integration separately, but to do so we need an algorithm that can take arbitrary holonomic functions [?] as input. For this purpose we employ the HolonomicFunctions package [?, ?] in Mathematica. We demonstrate the strategy with the example of the original Beukers triple integral:

$$\int_0^1 \int_0^1 \int_0^1 \underbrace{\frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}}}_{=:\, f_n(x,y,z)} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1}$$

Since the integrations commute, we have in principle six permutations in which order we can perform the integrations, but due to the symmetry of x and y, there are actually just three. Experiments suggested that the following strategy is the most efficient: we start with the integration w.r.t. z and obtain the following set of linear operators

$$\begin{aligned}
&\big\{(x-1)x(y-1)D_x-(x-1)(y-1)yD_y+n(x-y),\\
&\ (n+1)(xy-1)S_n+2(x-1)x(y-1)y^2D_y-(x-1)xy(3ny-n-y+1),\\
&\ (y-1)^2y^2(xy-1)D_y^2-(y-1)y(2nxy^2-4ny+2n-3xy^2+3xy+y-1)D_y\\
&\qquad+(n^2xy^3-4n^2y^2+4n^2y-n^2-2nxy^3+3nxy^2-ny+xy^3-2xy^2+xy)\big\}
\end{aligned}$$

that annihilates the integral $\int_0^1 f_n(x,y,z)\,\mathrm{d}z$. Here S_n denotes the forward shift in n, and D_x denotes the partial derivation w.r.t. x. Since we encounter recurrences and differential equations of order larger than one, it is clear that we cannot apply the Almkvist-Zeilberger algorithm again, but we really need a more general one. In the second step, we perform integration w.r.t. x and obtain the set of operators

$$\begin{aligned}
&\big\{4(y-1)^2y^2D_y^2+(n+1)^2yS_n-(y-1)y(11ny-8n-9y+4)D_y\\
&\qquad+(8n^2y^2-13n^2y+4n^2-6ny^2+5ny+2y^2-2y),\\
&\ 4(y-1)yS_nD_y-(ny+4n-3y+8)S_n-5(y-1)y^2D_y+y(8ny-3n-2y+2),\\
&\ 4(n+2)^2yS_n^2+\big((17n^2+51n+38)y^2-20(2n+3)^2y+16(2n+3)^2\big)S_n\\
&\qquad-(y-1)y^2(27ny-32n+42y-52)D_y\\
&\qquad+y(40n^2y^2-61n^2y+16n^2+50ny^2-72ny+12n-20y^2+44y-24)\big\}
\end{aligned}$$

that annihilates now the double integral $\int_0^1\int_0^1 f_n(x,y,z)\,\mathrm{d}z\,\mathrm{d}x$. Finally, we input the above set of operators once again to a holonomic creative telescoping algorithm, in order to perform the integration w.r.t. y, and this yields the single operator

$$(n+2)^3S_n^2-(2n+3)(17n^2+51n+39)S_n+(n+1)^3$$

that represents the second-order recurrence for the Beukers integral (1).

Now let us turn to the generalized Beukers triple integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^{a_1}(1-x)^{a_2} y^{b_1}(1-y)^{b_2} z^{c_1}(1-z)^{c_2}}{(1-z+xyz)^d} \cdot f_n(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2}$$

It can be treated in the same fashion as (1), with the major difference that the existence of up to seven additional parameters tremendously blows up all the expressions appearing in this calculation.

For general parameters $a_1, a_2, b_1, b_2, c_1, c_2, d$, the integral (2) does not satisfy a second-order linear recurrence, as desired, but only a third-order one. One can see this quickly by substituting the parameters by "random" integers (or even better: rational numbers!). With some effort (using evaluation/interpolation techniques, see below) we were also able to obtain the third-order recurrence for purely symbolic parameters $a_1, a_2, b_1, b_2, c_1, c_2$ in the case $d = 0$. The four coefficients of this recurrence are huge multivariate polynomials so that the whole recurrence consumes about 1 GB of memory.

One possible strategy that we considered was to extract conditions on the parameters, under which the order of this recurrence would drop, for example, by studying the zero set of the leading resp. trailing coefficient. Unfortunately, this did not reveal much useful information.

Instead, we performed a trial-and-error driven search for particular choices of the parameters, such that the resulting recurrence is of order two. Starting with one-, two-, and three-parameter families, we ended up with a (possibly infinite) family of six-parameter families, which we believe is the most general integral of the form (2) still giving a second-order recurrence. More precisely, we set

$$a_1 = b, \quad a_2 = c - f, \quad b_1 = e, \quad b_2 = a + f + i, \quad c_1 = a, \quad c_2 = c, \quad d = d, \tag{3}$$

and let a, b, c, d, e, f be arbitrary (i.e., symbolic) parameters, while i must be a nonnegative integer. That means, for each integer $i \geqslant 0$, we expect a second-order recurrence in the symbolic six parameters a, b, c, d, e, f. We have concretely computed such second-order six-parameter recurrences for $i = 0$, $i = 1$, and $i = 2$, and observed that the coefficient degrees in the parameters were growing linearly with i (see Table 1), suggesting that there is no second-order recurrence for symbolic i. Based on experimental results, we conjecture that for each nonnegative integer i a second-order recurrence exists, but we are unable to prove it in general.

Constructing these recurrences is quite a computational challenge. None of the creative telescoping algorithms [**?**, **?**] implemented in the HolonomicFunctions package was able to compute them directly with all parameters symbolic. Instead, we chose a hybrid strategy and treated three of the parameters (specifically: c, d, f) with evaluation/interpolation, while keeping the others (a, b, e) symbolic. This allowed us to circumvent rational function reconstruction and thus keep the number of necessary evaluation points low (namely: degree of the polynomial plus one, plus one for safety, instead of twice the polynomial degree

	(a,b,c,d,e,f,n)-degree	points	time/pt	total time	size
$i=0$	$(6,6,10,6,6,8,13)$	960	170 s	45 h + 0.5 h	18 M
$i=1$	$(7,7,12,7,7,10,15)$	1512	300 s	126 h + 3 h	47 M
$i=2$	$(8,8,14,8,8,12,17)$	2240	700 s	18 d + 8 h	106 M

Table 1: Computational data for three instances of a six-parameter family with second-order recurrence. The second column displays the coefficient degree of the output recurrence, the column "points" shows the number of evaluation points, "time/pt" refers to the time spend for each point. The next column gives the total timing (time spent in the evaluation phase + time for interpolation), and the last column shows the size (in MegaByte, according to Mathematica's ByteCount) of the recurrence with factored coefficients (keeping them in expanded form increases the size by a factor of about 3).

plus two, for rational reconstruction). Also we observed that there was no significant expression swell with respect to the numeric coefficients and therefore we did the computation without any modular arithmetic and chinese remaindering. Table 1 shows the increasing difficulty as the parameter i grows; we were not able (resp. not willing) to construct a second-order recurrence for $i > 2$. The recurrence that was mainly used in the rest of this paper is the five-parameter family that is obtained from (3) by setting $f = i = 0$.