Graphic and Symbolic Experiments with the Harry Dym Equation

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In fond memory of Harry Dym (Jan. 26, 1938 - July 18, 2024)

Abstract: We use both symbolic and numeric computations to experiment with the Harry Dym equation, of central importance in soliton theory.

Preface

On July 18, 2024, our beloved *academic father*, Harry Dym, suddenly died. That was such a shock, since Harry was so much alive, both physically and mathematically. This modest tribute to his memory is about the **Harry Dym equation**, one of the most important differential equations in *soliton theory.* Curiously, Harry never published anything about it, and the name was coined by the great soliton expert Martin Kruskal. In Harry's own words ([D1], p.10):

"Martin Kruskal spent the academic year 1973-1974 on sabbatical at the Weizmann Institute. In the course of the year he gave a number of interesting lectures on isospectral problems for the Schroedinger equation and the connections with the KdV equation. After so many years of living with the string equation, it seemed natural to explore analogous questions in this setting also. A few hours of calculations (and miscalculations) led to the conclusion that if the density of the string is parametrized by t and allowed to evolve according to the partial differential equation $r_t - r^3 r_{xxx} = 0$*, then the eigenvalues of the string equation* $-r^{-1}y'' = \lambda y$*. with appropriate side conditions, would stay fixed in time. In 1974, Martin reported on these calculations in a series of lectures at the Batelle Institute and called the PDE the Harry Dym equation. The name stuck, even though I never wrote any papers on the subject. (Actually a draft of a paper which explored a number of questions related to the theory of such equations was prepared in collaboration with Martin. But Martin took it back with him to Princeton, where it is presumably still collecting dust in his office.)"*

Since we like *symbolic* computation, we were wondering whether we can try to find as many as possible terms in the 'series expansion' of a generic solution of the Harry Dym equation subject to general *initial conditions* (in x). We were also curious how a typical solution looks near the origin of the (x, t)-plane. To that end one, of us (DZ) wrote a Maple package `HDequation.txt`, available from:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/HDequation.txt` .

Here we will describe some of the output produced by the *other 'one of us'* (SBE).

Series Expansions for the general solution of the Harry Dym Equation

Theorem 1: Let $a_0(t)$, $a_1(t)$, and $a_2(t)$ be *any* functions, or formal power series, in t. We are interested in the *series expansion*, in x, of the *general solution* $u(x,t)$, of the celebrated Harry Dym equation

$$u_t = u^3 u_{xxx} \quad ,$$

subject to the initial conditions

$$u(0,t) = a_0(t) \quad , \quad u_x(0,t) = a_1(t) \quad , \quad u_{xx}(0,t) = a_2(t) \quad .$$

Write

$$u(x,t) = \sum_{i=0}^{\infty} A_i(t) x^i \quad .$$

For $i \in \{0,1,2\}$, and any $r \geq 0$, let's abbreviate:

$$b_{i,r} := \frac{d^r}{dt^r} a_i(t) \quad .$$

We have:

$$A_0(t) = b_{0,0} \quad , \quad A_1(t) = b_{1,0} \quad , \quad A_2(t) = b_{2,0} \quad , \quad A_3(t) = \frac{b_{0,1}}{6b_{0,0}^3} \quad , \quad A_4(t) = \frac{b_{0,0}b_{1,1} - 3b_{0,1}b_{1,0}}{24b_{0,0}^4} \quad ,$$

$$A_5(t) = \frac{b_{0,0}^2 b_{2,1} - 3b_{0,0}b_{0,1}b_{2,0} - 3b_{0,0}b_{1,0}b_{1,1} + 6b_{0,1}b_{1,0}^2}{60b_{0,0}^5} \quad ,$$

$$A_6(t) = \frac{-18b_{0,0}^3 b_{1,0} b_{2,1} - 18b_{0,0}^3 b_{1,1} b_{2,0} + 72b_{0,0}^2 b_{0,1} b_{1,0} b_{2,0} + 36b_{0,0}^2 b_{1,0}^2 b_{1,1} - 60b_{0,0} b_{0,1} b_{1,0}^3 + b_{0,0} b_{0,2} - 6b_{0,1}^2}{720b_{0,0}^7} \quad .$$

$$A_7(t) = \frac{1}{5040b_{0,0}^8} \cdot$$

$$(-72b_{0,0}^4 b_{2,0} b_{2,1} + 144b_{0,0}^3 b_{0,1} b_{2,0}^2 + 144b_{0,0}^3 b_{1,0}^2 b_{2,1} + 288b_{0,0}^3 b_{1,0} b_{1,1} b_{2,0} - 720b_{0,0}^2 b_{0,1} b_{1,0}^2 b_{2,0}$$
$$-240b_{0,0}^2 b_{1,0}^3 b_{1,1} + 360b_{0,0} b_{0,1} b_{1,0}^4 + b_{0,0}^2 b_{1,2} - 21b_{0,0} b_{0,1} b_{1,1} - 15b_{0,0} b_{0,2} b_{1,0} + 105b_{0,1}^2 b_{1,0}) \quad ,$$

$$A_8(t) = \frac{1}{40320b_{0,0}^9} \cdot$$

$$(1440b_{0,0}^4 b_{1,0} b_{2,0} b_{2,1} + 720b_{0,0}^4 b_{1,1} b_{2,0}^2 - 3600b_{0,0}^3 b_{0,1} b_{1,0} b_{2,0}^2 - 1200b_{0,0}^3 b_{1,0}^3 b_{2,1} - 3600b_{0,0}^3 b_{1,0}^2 b_{1,1} b_{2,0}$$
$$+7200b_{0,0}^2 b_{0,1} b_{1,0}^3 b_{2,0} + 1800b_{0,0}^2 b_{1,0}^4 b_{1,1} - 2520b_{0,0} b_{0,1} b_{1,0}^5 + 2b_{0,0}^3 b_{2,2} - 78b_{0,0}^2 b_{0,1} b_{2,1} - 66b_{0,0}^2 b_{0,2} b_{2,0} - 21b_{0,0}^2 b_{1,0} b_{1,2}$$
$$-21b_{0,0}^2 b_{1,1}^2 + 462b_{0,0} b_{0,1}^2 b_{2,0} + 501b_{0,0} b_{0,1} b_{1,0} b_{1,1} + 177b_{0,0} b_{0,2} b_{1,0}^2 - 1416b_{0,1}^2 b_{1,0}^2) \quad .$$

To see $A_i(t)$ for $i \leq 30$, see the output file:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHDequation1.txt` .

If we take $a_0(t) = 1$, i.e. $u(0,t) = 1$ for all t, things simplify, and we have the following theorem.

Theorem 1': Let $a_1(t)$, and $a_2(t)$ be *any* functions, or formal power series, in t. We are interested in the *series expansion*, in x, of the *general solution* $u(x,t)$, of the celebrated Harry Dym equation

$$u_t = u^3 u_{xxx} \quad ,$$

subject to the initial conditions

$$u(0,t) = 1 \quad , \quad u_x(0,t) = a_1(t) \quad , \quad u_{xx}(0,t) = a_2(t) \quad .$$

Write

$$u(x,t) = \sum_{i=0}^{\infty} B_i(t) x^i \quad .$$

For $i \in \{1,2\}$, and any $r \geq 0$, let's abbreviate:

$$b_{i,r} := \frac{d^r}{dt^r} a_i(t) \quad .$$

We have:

$$B_0(t) = 1 \quad , \quad B_1(t) = b_{1,0} \quad , \quad B_2(t) = b_{2,0} \quad , \quad B_3(t) = 0 \quad , \quad B_4(t) = \frac{b_{1,1}}{24} \quad , \quad B_5(t) = -\frac{b_{1,0}b_{1,1}}{20} + \frac{b_{2,1}}{60} \quad ,$$

$$B_6(t) = \frac{1}{20} b_{1,1} b_{1,0}^2 - \frac{1}{40} b_{1,1} b_{2,0} - \frac{1}{40} b_{2,1} b_{1,0} \quad ,$$

$$B_7(t) = \frac{1}{5040} b_{1,2} - \frac{1}{21} b_{1,1} b_{1,0}^3 + \frac{2}{35} b_{1,1} b_{1,0} b_{2,0} + \frac{1}{35} b_{2,1} b_{1,0}^2 - \frac{1}{70} b_{2,1} b_{2,0} \quad ,$$

$$B_8(t) =$$

$$-\frac{1}{1920} b_{1,1}^2 - \frac{1}{1920} b_{1,0} b_{1,2} + \frac{1}{20160} b_{2,2} + \frac{5}{112} b_{1,1} b_{1,0}^4 - \frac{5}{56} b_{1,1} b_{1,0}^2 b_{2,0} - \frac{5}{168} b_{2,1} b_{1,0}^3 + \frac{1}{28} b_{1,0} b_{2,1} b_{2,0} + \frac{1}{56} b_{1,1} b_{2,0}^2 \quad ,$$

$$B_9(t) = \frac{1}{1120} b_{1,2} b_{1,0}^2 - \frac{1}{6720} b_{2,2} b_{1,0} + \frac{1}{560} b_{1,1}^2 b_{1,0} - \frac{1}{3360} b_{1,2} b_{2,0} - \frac{1}{2240} b_{1,1} b_{2,1} - \frac{1}{24} b_{1,1} b_{1,0}^5$$

$$+ \frac{5}{168} b_{2,1} b_{1,0}^4 + \frac{1}{84} b_{2,1} b_{2,0}^2 + \frac{5}{42} b_{1,1} b_{1,0}^3 b_{2,0} - \frac{5}{84} b_{1,0}^2 b_{2,1} b_{2,0} - \frac{5}{84} b_{1,0} b_{1,1} b_{2,0}^2 \quad ,$$

$$B_{10}(t) = -\frac{1}{11200} b_{2,1}^2 + \frac{1}{3628800} b_{1,3} + \frac{37}{22400} b_{1,1} b_{1,0} b_{2,1} + \frac{73}{67200} b_{1,2} b_{1,0} b_{2,0} - \frac{1}{24} b_{1,0} b_{2,1} b_{2,0}^2 - \frac{7}{48} b_{1,1} b_{1,0}^4 b_{2,0}$$

$$+ \frac{1}{12} b_{1,0}^3 b_{2,1} b_{2,0} + \frac{1}{8} b_{1,0}^2 b_{1,1} b_{2,0}^2 - \frac{2}{1575} b_{1,2} b_{1,0}^3 + \frac{73}{67200} b_{1,1}^2 b_{2,0}$$

$$-\frac{2}{525} b_{1,1}^2 b_{1,0}^2 + \frac{19}{67200} b_{2,2} b_{1,0}^2 - \frac{1}{11200} b_{2,2} b_{2,0} + \frac{7}{180} b_{1,1} b_{1,0}^6 - \frac{7}{240} b_{2,1} b_{1,0}^5 - \frac{1}{72} b_{1,1} b_{2,0}^3 \quad .$$

For expressions for $B_i(t)$ for all $i \leq 40$, see the output file:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHDequation1S.txt` .

For the analogous expansions for the solutions of the KortewegDe Vries (KdV) equation see the output files https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHDequation2.txt ,

and

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oHDequation2S.txt .

Pictures

It turns out that for most specific, random, choices of initial conditions $a_0(t)$, $a_1(t)$, $a_2(t)$, the above *formal power series* **converge** near the origin. So by truncating the series up to 30 terms, we got nice plots.

See

https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/HDequationPics.html .

Comment: Surpsingly (and perhaps not, we are not experts), the plots of the solutions for random initial conditions for the KdV and the HD equations look very similar, at least near the origin.

Appendix: Harry Dym's favorite quotes

Harry Dym was not *just* a great researcher, he was an amazing *teacher*, and *expositor*, and had a great sense of humor. Each of the chapters of his brilliant textbook *Linear Algebra in Action* [D2], that we **strongly recommend**, has a witty and sometimes funny *motto*. In order to encourage you to read that masterpiece, here the first eleven of them. For the remaining 15 read the book!

• Preface: *A foolish consistency is the hobgoblin of little minds, ...* - Ralph Waldo Emerson, *Self Reliance*

• Chapter 1:

The road to wisdom? Well it's plain and simple to express.

Err and err and err again, but less and less and less. (cited in *Mathematical Writing* by D.E. Knuth, T. Larrabee, and P.M. Roberts.)

• Chapter 2:

... People can tell you... do it like this, But that ain't the way to learn, You got to do it all by yourself. — Willie Mays

• Chapter 3: *I was working on the proof of one of my poems all morning, and took out a comma, In the afternoon I put it back again.* - Oscar Wilde.

• Chapter 4: *Can you imagine a mathematician writing* **Moby Dick***? Let my name be Ishmael, let the captain's name be Ahab, let the boats name be Pequod, and let the whale's name be as in the title* - B. A. Cipra

• Chapter 5: *Look at him, he doesn't drink, he doesn't smoke, he doesn't chew, he doesn't stay out late, and he still can't hit.* - Casey Stengel.

• Chapter 6: *Some people believe that football is a matter of life and death. I'm very disappointed with that attitude, I can assure you its much, much more important than that.* - Bill Shankly, former manager of Liverpool.

• Chapter 7 :

I give you now Professor Twist, A conscientious scientist,

. . . . . .

Camped on a tropic riverside, One day he missed his loving bride.

She had, the guide informed him later, Been eaten by an alligator.

Professor Twist could not but smile. "You mean," he said, " a crocodile."

The Purist, by Ogden Nash.

• Chapter 8 : *A proof should be as simple as possible, but not simpler.* - Paraphrase of Albert Einstein's remark on deep truths.

• Chapter 9 : *Not everything that one thinks, should one say; not everything that one says, should one write, and not everything that one writes, should one publish* - Dictum of the Soloveitchick family.

• Chapter 10 : *Let's throw everything away. Then there will be room for what's left* - Irene Dym.

References

[D1] Harry Dym, *Looking back*, in: *"Interpolation Theory, Systems Theory and Related Topics"*, The Harry Dym Anniversary volume, Operator Theory Advances and Applications **134** (2002), (Daniel Alpay, Israel Gohberg, and Victor Vinnikov, editors), 1-19.
`https://sites.math.rutgers.edu/~zeilberg/akherim/HarryDym60.pdf` .

[D2] Harry Dym, *"Linear Algebra in Action"* (Graduate Studies in Mathematics **78**), American Mathematical Society, 2007.

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